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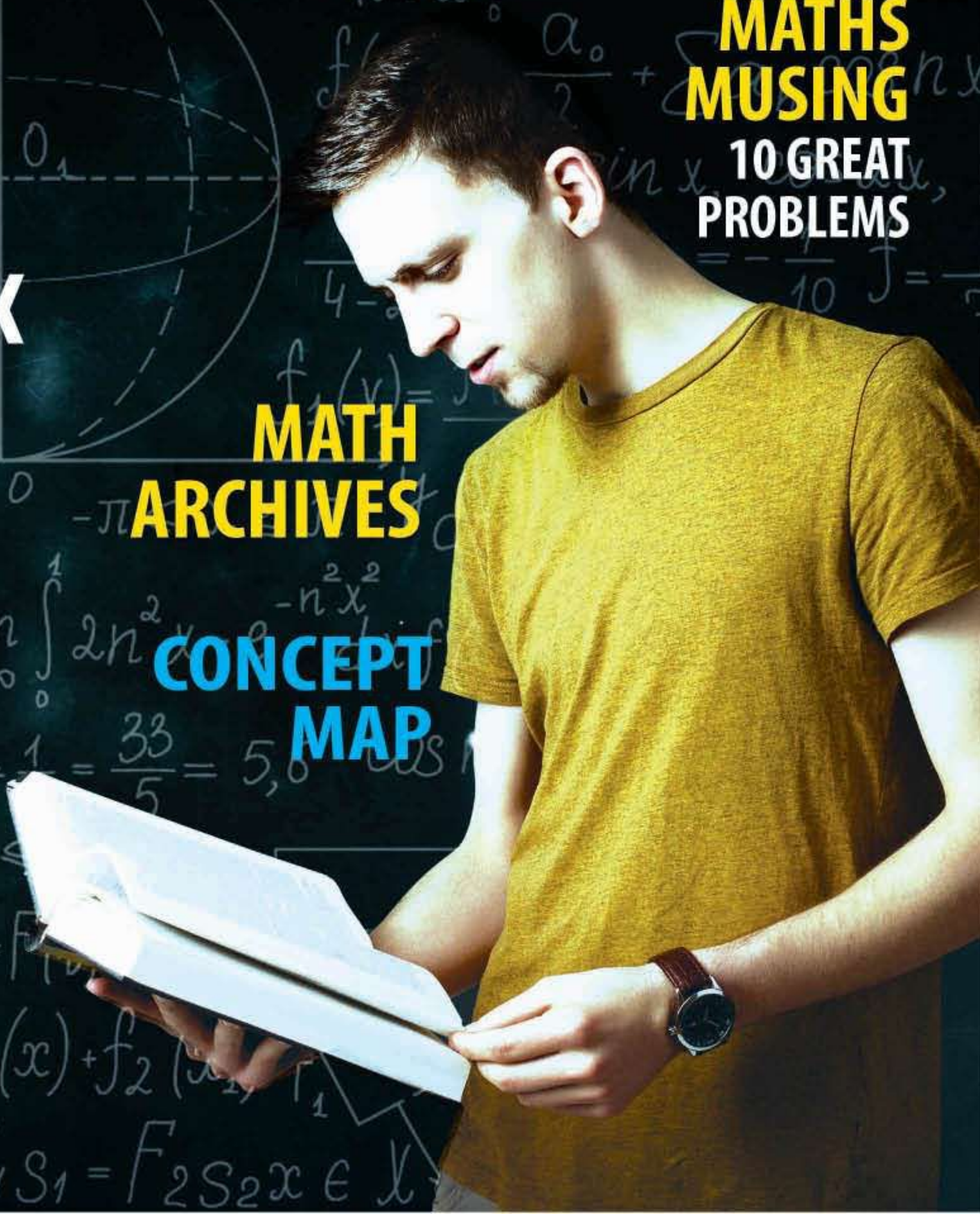

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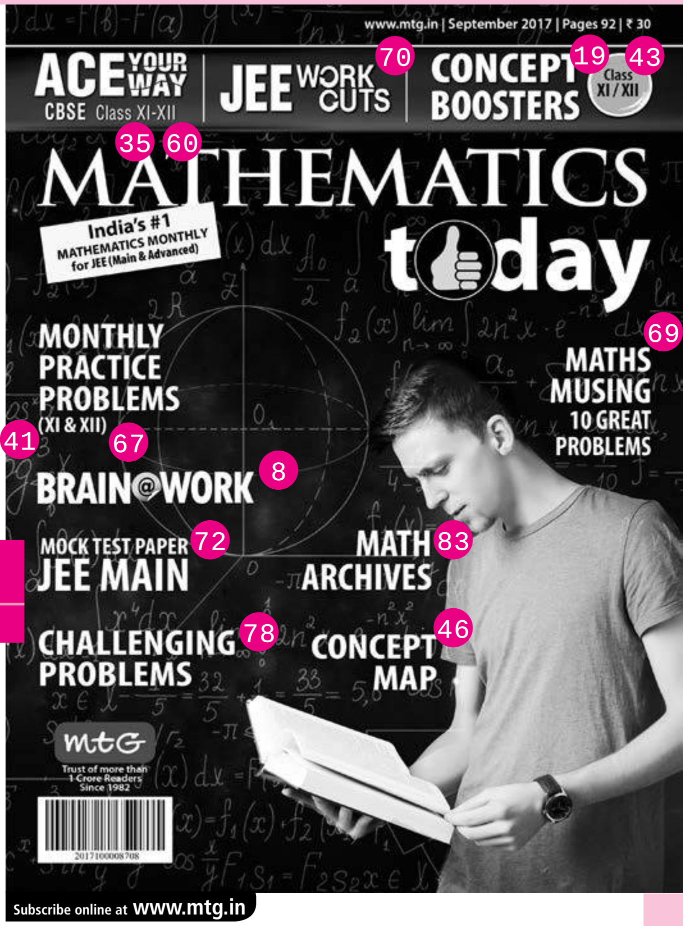
Corporate Office:
Plot 99, Sector 44 Institutional Area,
Gurgaon -122 003 (HR), Tel : 0124-6601200
e-mail : info@mtg.in website : www.mtg.in

Regd. Office:
406, Taj Apartment, Near Safdarjung Hospital,
Ring Road, New Delhi - 110029.
Managing Editor : Mahabir Singh
Editor : Anil Ahlawat



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BRAIN@WORK

TWO DIMENSIONAL GEOMETRY

1. The number of points on the line $3x + 4y = 5$, which are at a distance of $\sec^2\theta + 2\ \text{cosec}^2\theta$, $\theta \in R$, from the point (1, 3), is
(a) 1 (b) 4
(c) 3 (d) none of these

2. Let $ax + by + c = 0$ be a variable straight line, where a, b and c are 1st, 3rd and 7th terms of an increasing A.P. respectively. Then the variable straight line always passes through a fixed point which lies on
(a) $y^2 = 4x$ (b) $x^2 + y^2 = 5$
(c) $3x + 4y = 9$ (d) $x^2 + y^2 = 13$

3. The vertices of a triangle are $A(-1, -7)$, $B(5, 1)$ and $C(1, 4)$. The equation of the bisector BD of $\angle ABC$ is
(a) $x + 7y + 2 = 0$ (b) $x - 7y + 2 = 0$
(c) $x - 7y - 2 = 0$ (d) $x + 7y - 2 = 0$

4. The equation of the base of an equilateral triangle ABC is $x + y = 2$ and the vertex is $(2, -1)$. The area of the triangle ABC is (sq. units)
(a) $\frac{\sqrt{2}}{6}$ (b) $\frac{\sqrt{3}}{6}$ (c) $\frac{\sqrt{3}}{8}$ (d) none of these

5. The line $3x - 4y + 7 = 0$ is rotated through an angle $\pi/4$ in the clockwise direction about the point $(-1, 1)$. The equation of the line in its new position is
(a) $7y + x - 6 = 0$ (b) $7y - x - 6 = 0$
(c) $7y + x + 6 = 0$ (d) $7y - x + 6 = 0$

6. The equation of a line through the point (1, 2) whose distance from the point (3, 1) has the least possible value is
(a) $x + 2y = 3$ (b) $y = 2x$
(c) $y = x + 1$ (d) $x + 2y = 5$

7. Equation of straight line $ax + by + c = 0$, where $3a + 4b + c = 0$, which is at least distance from $(1, -2)$ is
(a) $3x + y - 17 = 0$ (b) $4x + 3y - 24 = 0$
(c) $3x + 4y - 25 = 0$ (d) $x + 3y - 15 = 0$

8. The curve $y = \left[\frac{1}{6}x^2 - 2x + 27\right]$ where $[x]$ is the greatest integer less than or equal to x, $6 \le x \le 9$ represents
(a) a parabola (b) part of the parabola
(c) a straight line (d) two straight line segments

9. Equation of circle through intersection of $x^2 + y^2 + 2x = 0$ and $x - y = 0$, having minimum radius is
(a) $x^2 + y^2 - 1 = 0$
(b) $x^2 + y^2 - x - y = 0$
(c) $x^2 + y^2 - 2x - 2y = 0$
(d) none of these

10. The common chord of the circle $x^2 + y^2 + 6x + 8y - 7 = 0$ and a circle passing through the origin, and touching the line $y = x$, always passes through the point
(a) $(-1/2, 1/2)$ (b) $(1, 1)$
(c) $(1/2, 1/2)$ (d) none of these

11. Three equal circles each of radius r touch one another. The radius of the circle touching all the three given circles internally is
(a) $(2+\sqrt{3})r$ (b) $\frac{(2+\sqrt{3})}{\sqrt{3}}r$
(c) $\frac{(2-\sqrt{3})}{\sqrt{3}}r$ (d) $(2-\sqrt{3})r$

Sanjay Singh Mathematics Classes, Chandigarh, Ph : 9888228231, 9216338231

Best of Luck

for your Journey Ahead

JEE Achievers - 2017

Keshav Gupta

MIT, USA(2017-21)
JEE Adv. AIR-269
JEE Main AIR-232

I am Keshav Gupta, student at KCS Educate during the sessions 2015-16 and 2016-17. The teachers at KCS Educate constantly motivated me and guided me towards the target. In the classroom, the focus always remained on keeping the students interested in the topics. Apart from academics, great emphasis was layed on motivational support for students. It has truly been a catalyst for success.

Keshav
Keshav Gupta

Tushar Agrawal

JEE Adv. AIR-609
JEE Main C.G. Topper
KVPY Qualified

I am Tushar Agrawal. I joined KCS for Two year program. The lectures delivered by Sir were very interacting and motivating. It never felt to me that "I need to study" but like a pleasure flow and eventually I began to enjoy studies. Further the tests and assignments helped me to understand my weak areas and my speed. Avnish Sir motivated at each stage and was ready to help at any time. I want to thank Avnish Sir and his team for their support.

Tushar

12. The set of points $P(x, y)$ such that their distance from (3, 0) is $\sqrt{2}$ times their distance from (0, 2) form a circle
(a) whose radius is greater than 5 units
(b) whose radius is equal to 5 units
(c) whose radius is less than 5 units
(d) none of these

13. If the circles $ax^2 + ay^2 + 2bx + 2cy = 0$ and $Ax^2 + Ay^2 + 2Bx + 2Cy = 0$ touch each other, then
(a) $bC = cB$ (b) $aC = cA$
(c) $aB = bA$ (d) $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$

14. Let $\phi(x , y) = 0$ be the equation of a circle. If $\phi(0, \lambda) = 0$ has equal roots $\lambda = 2, 2$ and $\phi(\lambda, 0) = 0$ has roots $\lambda = \frac{4}{5}, 5$ then the centre of the circle is
(a) (2, 29/10) (b) (29/10, 2)
(c) (– 2, 29/10) (d) none of these

15. The minimum distance between the circle $x^2 + y^2 = 9$ and the curve $2x^2 + 10y^2 + 6xy = 1$ is
(a) $2\sqrt{2}$ (b) 2
(c) $3-\sqrt{2}$ (d) $3-\frac{1}{\sqrt{11}}$

16. If one end of the diameter of a circle which touches x-axis is (3, 4) then the locus of other end of the diameter of the circle is/an
(a) parabola (b) hyperbola
(c) ellipse (d) circle

17. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then
(a) $d^2 + (3b - 2c)^2 = 0$
(b) $d^2 + (3b + 2c)^2 = 0$
(c) $d^2 + (2b - 3c)^2 = 0$
(d) $d^2 + (2b + 3c)^2 = 0$

18. The equation of the tangent to the parabola $y = (x - 3)^2$ parallel to the chord joining the points (3, 0) and (4, 1) is
(a) $2x - 2y + 6 = 0$ (b) $2y - 2x + 6 = 0$
(c) $4y - 4x + 13 = 0$ (d) $4x + 4y = 13$

19. The abscissa and ordinate of the end points A and B of a focal chord of the parabola $y^2 = 4x$ are respectively the roots of $x^2 - 3x + a = 0$ and $y^2 + 6y + b = 0$. The equation of the circle with AB as diameter is
(a) $x^2 + y^2 - 3x + 6y + 3 = 0$
(b) $x^2 + y^2 - 3x + 6y - 3 = 0$
(c) $x^2 + y^2 + 3x + 6y - 3 = 0$
(d) $x^2 + y^2 - 3x - 6y - 3 = 0$

20. AB is a chord of the parabola $y^2 = 4ax$ with vertex at A. BC is drawn perpendicular to AB meeting the axis at C. The projection of BC on the x-axis is
(a) a (b) $2a$ (c) $4a$ (d) $8a$

21. If normal at point P on parabola $y^2 = 4ax$, $(a > 0)$ meet it again at Q in such a way that OQ is of minimum length where O is vertex of parabola, then ΔOPQ is
(a) a right angled triangle
(b) obtuse angled triangle
(c) acute angled triangle
(d) none of these

22. If two distinct chords drawn from the point (4, 4) on the parabola $y^2 = 4ax$ are bisected on the line $y = mx$, then the set of value of m is given by
(a) $\left(\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right)$ (b) R
(c) $(0, \infty)$ (d) $(-2, 2)$

23. If t_1 and t_2 be the ends of a focal chord of the parabola $y^2 = 4ax$, then the equation $t_1x^2 + ax + t_2 = 0$ has
(a) imaginary roots
(b) both roots positive
(c) one positive and one negative root
(d) both roots negative

24. If the normal at three points $(ap^2, 2ap)$, $(aq^2, 2aq)$ and $(ar^2, 2ar)$ are concurrent then the common root of equations $px^2 + qx + r = 0$ and $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ is
(a) p (b) q (c) r (d) 1

25. The second degree equation $x^2 + 4y^2 + 2x + 16y + 13 = 0$ represent itself as
(a) a parabola (b) a pair of straight line
(c) an ellipse (d) a hyperbola

26. A tangent having a slope $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18} + \frac{y^2}{32} = 1$ intersects the major and minor axes at A and B. If O is the origin, then the area of ΔOAB is
(a) 48 sq. units (b) 9 sq. units
(c) 24 sq. units (d) 16 sq. units

27. Let P be a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci S_1 and S_2. If A be the area of ΔPS_1S_2, then the maximum value of A

(a) $ab\sin\theta$ (b) abe
(c) $a\sin\theta$ (d) $b\sin\theta$

28. Maximum distance of any point on the circle $(x-7)^2+(y-2\sqrt{30})^2=16$ from the centre of the ellipse $25x^2 + 16y^2 = 400$ is

(a) $\frac{1-\sqrt{3}}{2}$ (b) $\frac{3}{2}$
(c) 3 (d) none of these

29. If the eccentricity of the ellipse $\frac{x^2}{a^2+2}+\frac{y^2}{a^2+5}=1$ be $\frac{1}{\sqrt{3}}$, then length of latus rectum of ellipse is

(a) 4 (b) $\frac{18}{\sqrt{6}}$ (c) $\frac{10}{\sqrt{6}}$ (d) 8

30. The number of rational points on the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is

(a) infinite (b) 4
(c) 0 (d) 2

31. The radius of the circle passing through the points of intersection of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $x^2-y^2=0$ is

(a) $\frac{ab}{\sqrt{a^2+b^2}}$ (b) $\frac{\sqrt{2}\,ab}{\sqrt{a^2+b^2}}$
(c) $\frac{a^2-b^2}{\sqrt{a^2+b^2}}$ (d) $\frac{a^2+b^2}{\sqrt{a^2-b^2}}$

32. The pair of lines joining origin to the intersection of the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by the line $lx + my + n = 0$ are coincident if

(a) $a^2l^2 + b^2m^2 = n^2$ (b) $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{1}{n^2}$
(c) $\frac{l^2}{a^2}+\frac{m^2}{b^2}=n^2$ (d) none of these

33. The auxiliary circle of a family of ellipse passes through origin and makes intercept of 8 and 6 units on x-axis and y-axis respectively. If eccentricity of all such family is 1/2 then locus of focus will be

(a) $\frac{x^2}{16}+\frac{y^2}{9}=25$
(b) $4x^2 + 4y^2 - 32x - 24y + 75 = 0$
(c) $\frac{x^2}{16}-\frac{y^2}{9}=25$
(d) none of these

34. Equation of a common tangents to the curves $y^2 = 8x$ and $xy = -1$ is

(a) $3y = 9x + 2$ (b) $y = 2x + 1$
(c) $2y = x + 8$ (d) $y = x + 2$

35. If the foci of the ellipse $16x^2 + 7y^2 = 112$ and of the hyperbola $\frac{y^2}{144}-\frac{x^2}{a^2}=\frac{1}{25}$ coincide, then $a =$

(a) 3 (b) 9 (c) 81 (d) 8

36. The point of the curve $3x^2 - 4y^2 = 72$ which is nearest to the line $3x + 2y - 1 = 0$ is

(a) (6, 3) (b) (6, –3)
(c) (6, 6) (d) (6, 5)

37. Let PQ be a double ordinate of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If O be the centre of the hyperbola and OPQ is an equilateral triangle then the eccentricity e is

(a) $>\sqrt{3}$ (b) > 2
(c) $>\frac{2}{\sqrt{3}}$ (d) none of these

38. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is

(a) $9x^2 - 8y^2 + 18x - 9 = 0$
(b) $9x^2 - 8y^2 - 18x + 9 = 0$
(c) $9x^2 - 8y^2 - 18x - 9 = 0$
(d) $9x^2 - 8y^2 + 18x + 9 = 0$

39. The equation $\left|\sqrt{x^2+(y-1)^2}-\sqrt{x^2+(y+1)^2}\right| = K$ will represent a hyperbola for

(a) $K \in (0, 2)$ (b) $K \in (0, \infty)$
(c) $K \in (1, \infty)$ (d) none of these

40. $A(3, 4)$, $B(0, 0)$ and $C(3, 0)$ are vertices of ΔABC. If 'P' is a point inside ΔABC, such that $d(P, BC) <$ min.$\{d(P, AB), d(P, AC)\}$. Then maximum of $d(P, BC)$ is [$d(P, BC)$ represent distance between P and BC]

(a) 1 (b) 1/2
(c) 2 (d) none of these

41. If coordinates of the vertices of a triangle are (2, 0), (6, 0) and (1, 5) then distance between its orthocentre and circumcentre is
(a) 4 (b) 6
(c) 5 (d) none of these

42. If the vertices of a triangle are $A(1, 4)$, $B(3, 0)$ and $C(2, 1)$ then the length of the median passing through C is
(a) 1 (b) 2 (c) $\sqrt{2}$ (d) $\sqrt{3}$

43. The equation to the locus of a point which moves so that its distance from x-axis is always one half its distance from the origin is
(a) $x^2 + 3y^2 = 0$ (b) $x^2 - 3y^2 = 0$
(c) $3x^2 + y^2 = 0$ (d) $3x^2 - y^2 = 0$

44. A ray of light coming from the point (1, 2) is reflected at a point A on the x-axis and then passes through the point (5, 3). The coordinates of the point A are
(a) $\left(\frac{13}{5}, 0\right)$ (b) $\left(\frac{5}{13}, 0\right)$
(c) (–7, 0) (d) none of these

45. The locus of the point of intersection of the lines $x + 4y = 2a\sin\theta$, $x - y = a\cos\theta$ where θ is a variable parameter is
(a) $5x^2 + 20y^2 = a^2$ (b) $5x^2 + 20y^2 = 2a^2$
(c) $5x^2 + 20y^2 = 3a^2$ (d) $5x^2 + 20y^2 = 4a^2$

SOLUTIONS

1. (d) : The perpendicular distance of (1, 3) from the line $3x + 4y = 5$ is 2 units while,
$\sec^2\theta + 2\operatorname{cosec}^2\theta \geq 3$ {as $\sec^2\theta, \operatorname{cosec}^2\theta \geq 1$}

2. (d) : Let the common difference of A.P. is d then $b = a + 2d$ and $c = a + 6d$, so variable straight line will be $ax + (a + 2d)y + a + 6d = 0$
$\Rightarrow a(x + y + 1) + d(2y + 6) = 0$
which always passes through (2, –3).

3. (b) : We have $AB = 10$, $BC = 5$.

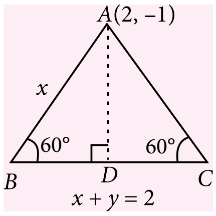
By bisector property $\frac{AD}{DC} = \frac{10}{5} = \frac{2}{1}$

$\Rightarrow$ Co-ordinates of D are $\left(\frac{1}{3}, \frac{1}{3}\right)$.

Hence equation of BD is

$$y - 1 = \left(\frac{(1/3) - 1}{(1/3) - 5}\right)(x - 5) \quad \text{or} \quad x - 7y + 2 = 0$$

4. (b) : Let side AB is x



$\therefore$ Length $AD = \frac{|2 - 1 - 2|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

In ΔABD, $\sin 60° = \frac{1}{\sqrt{2}x}$

$\frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}x} \Rightarrow x = \sqrt{\frac{2}{3}}$

$\therefore$ Area of an equilateral triangle

$= \frac{\sqrt{3}}{4}\frac{2}{3} = \frac{\sqrt{3}}{6}$ sq. units

5. (a) : As (–1, 1) is a point on $3x - 4y + 7 = 0$, the rotation is possible.

Slope of the given line $= \frac{3}{4}$

Slope of the line in its new position $= \frac{\frac{3}{4} - 1}{1 + \frac{3}{4}} = -\frac{1}{7}$

The required equation is $y - 1 = -\frac{1}{7}(x + 1)$
or $7y + x - 6 = 0$.

6. (b) : Line at least distance from (3, 1) points will be perpendicular to line joining given two points (1, 2) and (3, 1).
Slope of the line joining the points (1, 2) and (3, 1) is
$m = \frac{2 - 1}{1 - 3} = -\frac{1}{2}$
Hence the slope of the required line is 2 and its equation is $y - 2 = 2(x - 1)$.

7. (d) : It passes through a fixed point (3, 4)

Slope of line joining (3, 4) and (1, – 2) is $\frac{-6}{-2} = 3$

$\therefore$ Slope of required line = –1/3

Equation is $y - 4 = -\frac{1}{3}(x - 3)$

$\Rightarrow x + 3y - 15 = 0$

8. (d) : Given, $y = \left[\frac{1}{6}x^2 - 2x + 27\right]$

Let $f(x) = \frac{1}{6}x^2 - 2x + 27$ and $f'(x) \geq 0$ in given interval $6 \leq x \leq 9$. Hence range of $f(x) = [f(6), f(9)]$.
For $x \in [6, 9]$, y take only two integral values.

9. (d) : Equation of required circle is
$x^2 + y^2 + 2x + \lambda(x - y) = 0$

whose centre of circle is $\left(-\left(\frac{2 + \lambda}{2}\right), \frac{\lambda}{2}\right)$

and radius of circle is $\sqrt{\left(\frac{2+\lambda}{2}\right)^2+\frac{\lambda^2}{4}}=\sqrt{\frac{(\lambda+1)^2+1}{2}}$

Radius of circle is minimum when $\lambda = -1$.

Hence, required equation of circle is $x^2 + y^2 + x + y = 0$

10. (c): Let circle touching the line $y = x$ at $(0, 0)$ point is $S_1 : x^2 + y^2 + \lambda(x - y) = 0$ and

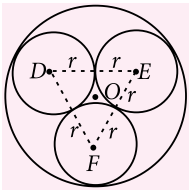
$S_2 : x^2 + y^2 + 6x + 8y - 7 = 0$

$\therefore$ Common chord is $S_1 - S_2 = 0$

$\Rightarrow \lambda(x - y) - 6x - 8y + 7 = 0$

$\Rightarrow x = y = 1/2$

11. (b) : $\because$ ΔDEF is an equilateral with side $2r$ if radius of circumcircle DEF is R_1, then



Area of $\Delta DEF = \frac{\sqrt{3}}{4}(2r)^2=\sqrt{3}r^2$

$$\sqrt{3}r^2=\frac{2r\cdot 2r\cdot 2r}{4R_1} \Rightarrow R_1=\frac{2r}{\sqrt{3}}$$

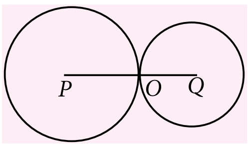
$\therefore$ Radius of the circle touching all the three given circles $=r+R_1=r+\frac{2r}{\sqrt{3}}=\frac{(2+\sqrt{3})r}{\sqrt{3}}$

12. (a) : The prescribed condition gives

$(x - 3)^2 + y^2 = 2\{x^2 + (y - 2)^2\}$

i.e., $x^2 + y^2 + 6x - 8y - 1 = 0$ which is a circle with centre $(-3, 4)$ and radius $=\sqrt{9+16+1}=\sqrt{26}>5$

13. (a) : Obviously both circles, pass through the origin and therefore O must be the point of contact.

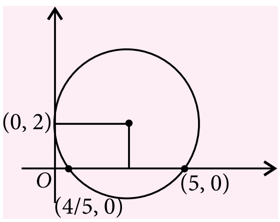


$\therefore$ Centres P, O, Q are collinear.

$\left(-\frac{b}{a},-\frac{c}{a}\right)$, $(0, 0)$ and $\left(-\frac{B}{A},-\frac{C}{A}\right)$ are collinear

$\therefore \frac{c}{b}=\frac{C}{B}$ or $cB=bC$

14. (b) : Given that circle touches y-axis at $(0, 2)$ point and intersects the x-axis at $(4/5, 0)$ and $(5, 0)$.



Centre: $\left(\frac{29}{10}, 2\right)$

15. (b) : Let $(r\cos\theta, r\sin\theta)$ be any point on the curve $2x^2 + 10y^2 + 6xy = 1$ then

$2r^2\cos^2\theta + 10r^2\sin^2\theta + 6r^2\sin\theta\cos\theta = 1$

Also, $r^2=\frac{1}{2+8\sin^2\theta+3\sin 2\theta}$

$=\frac{1}{6+3\sin 2\theta-4\cos 2\theta} \Rightarrow r_{max}=1$

Minimum distance between curve = 2.

16. (a) : Let other end is (h, k)

So centre $\equiv \left(\frac{h+3}{2},\frac{k+4}{2}\right)$ touching x-axis means $r=\frac{k+4}{2}$

So $\frac{k+4}{2}=\sqrt{\left(\frac{h+3}{2}-3\right)^2+\left(\frac{k+4}{2}-4\right)^2}$

Hence locus is $x^2 - 6x - 16y + 9 = 0$

Which is a parabola.

17. (d) : The two parabolas intersect at $(0, 0)$ and $(4a, 4a)$. The equation of their common chord must be $y = x$ which must be same as given line

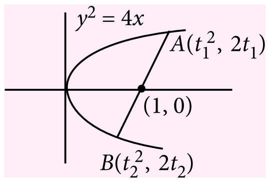
$2bx + 3cy + 4d = 0$

$\Rightarrow 2b = -3c,\ d = 0 \quad \Rightarrow (2b + 3c)^2 + d^2 = 0$

18. (c): $y' = 2(x - 3) = 1$ gives the point $\left(\frac{7}{2},\frac{1}{4}\right)$ and the required tangent is $y-\frac{1}{4}=1\left(x-\frac{7}{2}\right)$

or $4y - 4x + 13 = 0$.

19. (b) : $t_1t_2 = -1$ as AB is focal chord



$x^2 - 3x + a = 0$;

$x_1 + x_2 = 3$ and $x_1x_2 = a$

$y^2 + 6y + b = 0$;

$y_1 + y_2 = -6$ and $y_1y_2 = b$

$x_1x_2=\frac{1}{t_1^2}\cdot t_1^2=1=a$

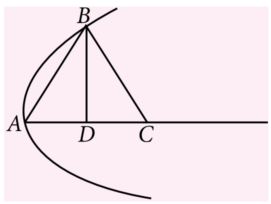
$y_1y_2=2t_1\left(-\frac{2}{t_1}\right)=-4=b$

$\therefore a = 1, b = -4$

Equation of circle with AB as diameter is

$x^2 + y^2 - 3x + 6y - 3 = 0$

20. (c): Let B be $(at^2, 2at)$



Slope of $AB = \frac{2}{t}$.

Equation of BC is

$$y-2at=-\frac{t}{2}(x-at^2)$$

This meets $y = 0$ at C whose x-coordinate $= 4a + at^2$

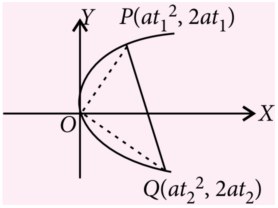
Also, $D = (at^2, 0)$

$\therefore DC = 4a + at^2 - at^2 = 4a$.

21. (a) : $\therefore$ Normal at $P(t_1)$ meets at $Q(t_2)$

$$t_2 = -\frac{2}{t_1} - t_1$$

$$|t_2| \geq 2\sqrt{2}$$



For minimum length of OQ, $|t_2|$ should be minimum

i.e. $|t_2| = 2\sqrt{2}$

If $t_2 = -2\sqrt{2} \Rightarrow t_1 = \sqrt{2}$

Slope of $OQ = \dfrac{2}{t_2} = m_1$ and of $OP = \dfrac{2}{t_1} = m_2$

$\therefore$ $m_1 m_2 = -1 \Rightarrow \Delta OPQ$ is right angled triangle.

22. (a) : Any point on the line $y = mx$ can be taken as (t, mt).

Equation of the chord of parabola with this as mid point

$$ymt - 2(x + t) = m^2t^2 - 4t$$

It passes through (4, 4)

$$4mt - 2(4 + t) = m^2t^2 - 4t$$

$\Rightarrow$ $m^2t^2 - 2(2m + 1)t + 8 = 0$

For two such chords, we must have $D > 0$

$\Rightarrow$ $(2m + 1)^2 - 8m^2 > 0$

$$4m^2 - 4m - 1 < 0 \Rightarrow \frac{1-\sqrt{2}}{2} < m < \frac{1+\sqrt{2}}{2}$$

23. (c): Product of roots < 0

24. (d) : Given points are co-normal points

$\Rightarrow$ $p + q + r = 0$

Common root is 1.

25. (c): $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 4 & 8 \\ 1 & 8 & 13 \end{vmatrix} = -16 \neq 0$

$\Rightarrow$ does not represent a pair of straight line.

Again $h^2 = 0$ and $ab = 4 \Rightarrow h^2 < ab$

$\Rightarrow$ An ellipse.

26. (c): Any point on the ellipse $\dfrac{x^2}{(3\sqrt{2})^2} + \dfrac{y^2}{(4\sqrt{2})^2} = 1$

can be taken as $(3\sqrt{2}\cos\theta, 4\sqrt{2}\sin\theta)$ and the slope of

the tangent $= -\dfrac{b^2x}{a^2y} = -\dfrac{32(3\sqrt{2}\cos\theta)}{18(4\sqrt{2}\sin\theta)} = -\dfrac{4}{3}\cot\theta$ ... (i)

Given slope of the tangent $= -\dfrac{4}{3}$ ... (ii)

From equations (i) and (ii), we get

$$\cot\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

Hence the equation of the tangent is

$$\frac{x \cdot \frac{1}{\sqrt{2}}}{3\sqrt{2}} + \frac{y \cdot \frac{1}{\sqrt{2}}}{4\sqrt{2}} = 1 \quad i.e. \quad \frac{x}{6} + \frac{y}{8} = 1$$

Hence $A = (6, 0)$, $B = (0, 8)$

Area of $\Delta OAB = \dfrac{1}{2} \times 6 \times 8 = 24$ sq. units

27. (b) : Let $P(a\cos\theta, b\sin\theta)$ be the variable point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Then,

A = Area of ΔPS_1S_2,

$$= \frac{1}{2}\begin{vmatrix} a\cos\theta & b\sin\theta & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix} = \frac{1}{2} b\sin\theta \times 2ae = abe\sin\theta$$

Area $= abe\sin\theta$, which is maximum when $\theta = \pi/2$

$\therefore$ $A_{max} = abe$

28. (d) : $\because$ Centre of ellipse (0, 0) and centre of circle is $(7, 2\sqrt{30})$ and radius is 4

$\therefore$ Maximum distance of any point on the circle from the centre of the ellipse

$$= \sqrt{(7-0)^2 + (2\sqrt{30} - 0)^2} + 4 = 17$$

29. (a) : Clearly $a^2 + 5 > a^2 + 2$

So $(a^2 + 2) = (a^2 + 5)\left(1 - \dfrac{1}{3}\right)$

$$3a^2 + 6 = 2a^2 + 10 \Rightarrow a^2 = 4$$

$\therefore$ Equation of ellipse is $\dfrac{x^2}{6} + \dfrac{y^2}{9} = 1$

$\therefore$ Length of latus rectum $= \dfrac{2 \times 6}{3} = 4$

30. (a) : $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Let any point on ellipse be $(3\cos\theta, 2\sin\theta)$

Since $\sin\theta$ and $\cos\theta$ can be rational for infinite many value of $\theta \in [0, 2\pi]$.

31. (b) : The circle through the points of intersection of the two curves will have centre at origin.

Solving $x^2 - y^2 = 0$ and $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we get

$$x^2 = y^2 = \frac{a^2b^2}{a^2 + b^2}$$

Therefore radius of circle $= \sqrt{\dfrac{2a^2b^2}{a^2+b^2}} = \dfrac{\sqrt{2}ab}{\sqrt{a^2+b^2}}$

32. (a) : $lx + my + n = 0 \Rightarrow \frac{lx+my}{-n}=1$

By Theory of Homogenization, we can get the pairs of line $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{lx+my}{-n}\right)^2$

$$\left(\frac{n^2}{a^2}-l^2\right)x^2+\left(\frac{n^2}{b^2}-m^2\right)y^2-2lmxy=0$$

This represent a pair of coincident lines if

$$l^2m^2-\left(\frac{n^2}{a^2}-l^2\right)\left(\frac{n^2}{b^2}-m^2\right)=0$$

$$\frac{n^4}{a^2b^2}=\frac{n^2m^2}{a^2}+\frac{n^2l^2}{b^2} \Rightarrow a^2l^2+b^2m^2=n^2$$

33. (b) : Centre is (4, 3) and distance of focus from centre is $ae=\frac{5}{2}$

$\therefore$ Locus is $(x-4)^2+(y-3)^2=\frac{25}{4}$

34. (d) : Equation of a tangent at $(at^2, 2at)$ to $y^2 = 8x$ is $ty = x + at^2$ where $4a = 8$ *i.e.* $a = 2$

$\Rightarrow$ $ty = x + 2t^2$ which intersects the curve $xy = -1$ at the points given by $\frac{x(x+2t^2)}{t}=-1$ clearly $t \neq 0$

or $x^2 + 2t^2x + t = 0$ and will be a tangent to the curve if the roots of this quadratic equation are equal, for which $4t^4 - 4t = 0 \Rightarrow t = 0$ or $t = 1$ and an equation of a common tangent is $y = x + 2$.

35. (b) : For the ellipse, $\frac{x^2}{7}+\frac{y^2}{16}=1$

$$e^2=\frac{a^2-b^2}{a^2}=\frac{9}{16} \Rightarrow e=\frac{3}{4}$$

$$\therefore\ ae=4\times\frac{3}{4}=3 \qquad \text{... (i)}$$

For the hyperbola $\frac{y^2}{\frac{144}{25}}-\frac{x^2}{\frac{a^2}{25}}=1$

$$e'^2=\frac{a^2+b^2}{a^2}=\frac{\frac{144}{25}+\frac{a^2}{25}}{\frac{144}{25}} \Rightarrow e'=\frac{\sqrt{144+a^2}}{12}$$

$$\therefore\ ae'=\frac{12}{5}\times\frac{\sqrt{144+a^2}}{12} \qquad \text{(ii)}$$

According to given condition,

$$3=\frac{12}{5}\times\frac{\sqrt{144+a^2}}{12} \Rightarrow 15=\sqrt{144+a^2}$$

$\Rightarrow a^2 = 225 - 144 = 81 \Rightarrow a = 9$

36. (b) : For the nearest point on the curve, tangent drawn to curve at that point should be parallel to the given line

$$\therefore\ 6x_1-8y_1\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)=\frac{3}{4}\cdot\frac{x_1}{y_1}=-\frac{3}{2}$$

$\Rightarrow x_1 = -2y_1$

which satisfy $3x^2 - 4y^2 = 72 \Rightarrow 12y_1^2 - 4y_1^2 = 72$

$\Rightarrow y_1^2 = 9 \Rightarrow y_1 = \pm 3 \Rightarrow x_1 = \pm 6$

Hence required points are $(-6, 3)$ and $(6, -3)$.

37. (c): Let P be (α, β) then $PQ = 2\beta$ and $OP=\sqrt{\alpha^2+\beta^2}$

Since OPQ is an equilateral triangle, $OP = PQ$

$$\alpha^2+\beta^2=4\beta^2=\alpha^2=3\beta^2 \Rightarrow \alpha=\pm\sqrt{3}\beta$$

$\because$ (α, β) is situated on the hyperbola

$$\therefore\ \frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}=1 \Rightarrow \frac{3\beta^2}{a^2}-\frac{\beta^2}{b^2}=1 \Rightarrow \frac{3}{a^2}-\frac{1}{b^2}=\frac{1}{\beta^2}>0$$

$$\frac{b^2}{a^2}>\frac{1}{3} \Rightarrow e^2-1>\frac{1}{3} \Rightarrow e^2>\frac{4}{3} \Rightarrow e>\frac{2}{\sqrt{3}}$$

38. (b) : $x = 9$ meets the hyperbola $x^2 - y^2 = 9$ at $(9, 6\sqrt{2})$ and $(9, -6\sqrt{2})$. The equations of the tangents to the hyperbola at these points are $3x-2\sqrt{2}y-3=0$ and $3x+2\sqrt{2}y-3=0$

$\therefore$ Joint equation of the two tangents is

$$(3x-2\sqrt{2}y-3)(3x+2\sqrt{2}y-3)=0$$

$$\Rightarrow (3x-3)^2-(2\sqrt{2}y)^2=0 \Rightarrow 9x^2-8y^2-18x+9=0.$$

39. (a) : We have, $\left|\sqrt{x^2+(y-1)^2}-\sqrt{x^2+(y+1)^2}\right|=K$

which is equivalent to $|S_1P - S_2P|$ = const.

Where $S_1 \equiv (0, 1)$, $S_2 \equiv (0, -1)$ and $P \equiv (x, y)$

Now, $2a = K$ [where $2a$ is the transverse axis and e is the eccentricity] and $2ae = S_1S_2 = 2$

Dividing, we have $e=\frac{2}{K}$

Since, $e > 1$ for a hyperbola, therefore $K < 2$

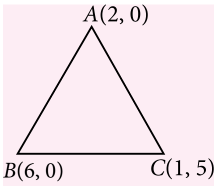
Also, K must be a positive quantity. Hence, we have, $K \in (0, 2)$.

40. (a) : The required point is point of intersection of internal angle bisectors

$\therefore$ $P(x, y)$ = incentre of Δ.

41. (c): Centroid $\equiv\left(\frac{2+6+1}{3},\frac{0+0+5}{3}\right)\equiv\left(3,\frac{5}{3}\right)$

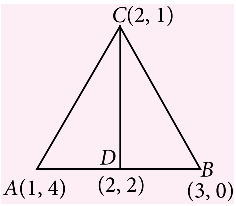
and for orthocentre equation of line perpendicular to AB passing through $C(1, 5)$ is
$x = 1$ ... (i)



Eq. of line perpendicular to AC and passing through
$B(6, 0)$ is $y = (x - 6)1/5$
$\Rightarrow 5y = x - 6$ ... (ii)
Solving (i) and (ii) we get
Orthocentre $\equiv (1, -1)$
We known centroid divides orthocentre and circumcentre in 2 : 1 (internally)
$\therefore$ Circumcentre is (4, 3)
$\therefore$ Distance between orthocentre and circumcentre

$$= \sqrt{(1-4)^2 + (-1-3)^2} = 5$$

42. (a) : Median $CD = \sqrt{(2-2)^2 + (2-1)^2} = 1$

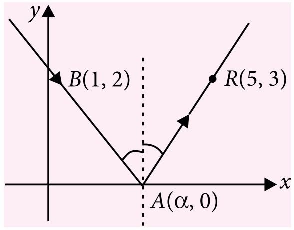


43. (b) : Let the point (h, k)

Given, $|k| = \frac{1}{2}\sqrt{h^2 + k^2}$

Squaring $4k^2 = h^2 + k^2 \Rightarrow h^2 - 3k^2 = 0$
$\therefore$ Locus $x^2 - 3y^2 = 0$

44. (a) : Let, the coordinates of point A is $(\alpha, 0)$
Now $-m_{AB} = m_{AR}$
If AR makes an angle θ with +ve x-axis, then AB makes $(\pi - \theta)$, therefore
$-m_{AB} = m_{AR}$



$$-\left(\frac{0-2}{\alpha-1}\right) = \left(\frac{0-3}{\alpha-5}\right)$$

$$\Rightarrow 2(\alpha-5) = -3(\alpha-1) \Rightarrow \alpha = \frac{13}{5}$$

$\therefore$ A is $\left(\frac{13}{5}, 0\right)$

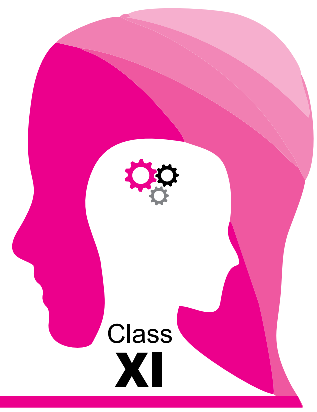
45. (d) : We have, $x + 4y = 2a\sin\theta$ ... (i)
$x - y = a\cos\theta$ .....(ii)

$$\frac{(x+4y)^2}{4} + (x-y)^2 = a^2 \quad \text{[From (i) and (ii)]}$$

$\Rightarrow 5x^2 + 20y^2 = 4a^2$

CONCEPT BOOSTERS

Complex Numbers



This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR, B.Tech, IIT Kanpur

- A number of the form $x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$, is called a complex number and 'i' is called iota.
  A complex number is usually denoted by z and the set of complex numbers is denoted by C.
  i.e., $C = \{x + iy : x \in R, y \in R, i = \sqrt{-1}\}$
  $5 + 3i$, $-1 + i$, $0 + 4i$, $4 + 0i$ etc. are complex numbers.
  - For any positive real number a, we have $\sqrt{-a} = \sqrt{-1 \times a} = \sqrt{-1}\sqrt{a} = i\sqrt{a}$
  - The property $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is valid only if at least one of a and b is non-negative.
- **Integral powers of iota (i) :** Since $i = \sqrt{-1}$ hence we have $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$. To find the value of $i^n (n > 4)$.
  Let $n = 4q + r$ where $0 \le r \le 3$,
  $\therefore \quad i^n = i^{4q+r} = (i^4)^q.(i)^r = (1)^q.(i)^r = i^r$
  In general we have the following results $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, where n is any integer.

REAL AND IMAGINARY PARTS OF A COMPLEX NUMBER

If x and y are two real numbers, then a number of the form $z = x + iy$ is called a complex number. Here 'x' is called the real part of z and 'y' is known as the imaginary part of z. The real part of z is denoted by Re(z) and the imaginary part by Im(z).
If $z = 3 - 4i$, then Re(z) = 3 and Im(z) = – 4.
A complex number z is purely real if its imaginary part is zero *i.e.*, Im(z) = 0 and purely imaginary if its real part is zero *i.e.*, Re(z) = 0.

ALGEBRAIC OPERATIONS WITH COMPLEX NUMBERS

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers then,

Addition $(z_1 + z_2) : (a + ib) + (c + id) = (a + c) + i(b + d)$
Subtraction $(z_1 - z_2) : (a + ib) - (c + id) = (a - c) + i(b - d)$
Multiplication $(z_1 \cdot z_2) : (a + ib)(c + id) = (ac - bd) + i(ad + bc)$
Division $(z_1/z_2) : \dfrac{a+ib}{c+id}$

(where at least one of c and d is non-zero)

$$= \frac{(a+ib)}{(c+id)} \cdot \frac{(c-id)}{(c-id)} = \frac{(ac+bd)}{c^2+d^2} + \frac{i(bc-ad)}{c^2+d^2}.$$

- **Properties of algebraic operations on complex numbers :** Let z_1, z_2 and z_3 are any three complex numbers then their algebraic operations satisfy following properties :
  - $z_1 + z_2 = z_2 + z_1$
  - $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
  - $z_1 z_2 = z_2 z_1$
  - $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.
  - $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$
  - $(z_2 + z_3) z_1 = z_2 z_1 + z_3 z_1$.

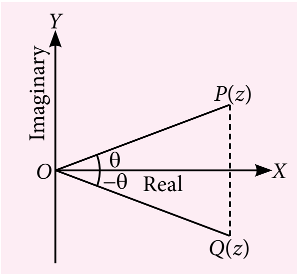
EQUALITY OF TWO COMPLEX NUMBERS

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if their real and imaginary parts are separately equal. *i.e.*, $z_1 = z_2$
$\Leftrightarrow \; x_1 + iy_1 = x_2 + iy_2$
$\Leftrightarrow \; x_1 = x_2$ and $y_1 = y_2$.
Complex numbers do not possess the property of order *i.e.*, $(a + ib) < (\text{or}) > (c + id)$ is not defined. For example, the statement $(9 + 6i) > (3 + 2i)$ makes no sense.

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91).
He trains IIT and Olympiad aspirants.

CONJUGATE OF A COMPLEX NUMBER

- **Conjugate complex number :** If there exists a complex number $z = a + ib$, $(a, b) \in R$, then its conjugate is defined as $\bar{z} = a - ib$.



Geometrically, the conjugate of z is the reflection or mirror image of z about real axis.

- **Properties of conjugate :** If z, z_1 and z_2 are existing complex numbers, then we have the following results:
  - $\overline{(\bar{z})} = z$
  - $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
  - $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

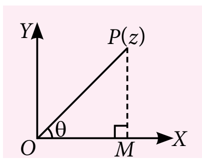
    In general, $\overline{z_1 \cdot z_2 \cdot z_3 .....z_n} = \bar{z}_1 \cdot \bar{z}_2 \cdot \bar{z}_3 .....\bar{z}_n$
  - $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{(\bar{z}_1)}{(\bar{z}_2)}, [z_2 \neq 0]$
  - $(\bar{z})^n = \overline{(z^n)}$
  - $z + \bar{z} = 2\text{Re}(z) = 2\text{Re}(\bar{z}) =$ purely real
  - $z - \bar{z} = 2i\text{Im}(z) =$ purely imaginary
  - $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\text{Re}(z_1\bar{z}_2) = 2\text{Re}(\bar{z}_1 z_2)$

Reciprocal of a complex number : For an existing non-zero complex number $z = a + ib$, the reciprocal is given by $z^{-1} = \dfrac{1}{z} = \dfrac{\bar{z}}{z \cdot \bar{z}} = \dfrac{\bar{z}}{|z|^2}$.

MODULUS OF A COMPLEX NUMBER

Modulus of a complex number $z = a + ib$ is defined by a positive real number given by $|z| = \sqrt{a^2 + b^2}$, where a, b are real numbers. Geometrically $|z|$ represents the distance of point P from the origin, *i.e.* $|z| = OP$.

If $|z| = 1$ the corresponding complex number is known as **unimodular complex number**. Clearly z lies on a circle of unit radius having centre (0, 0).



Properties of modulus

- $|z| \geq 0 \Rightarrow |z| = 0$ if $z = 0$ and $|z| > 0$ if $z \neq 0$.
- $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$
- $|z| = |\bar{z}| = |-z| = |-\bar{z}| = |zi|$
- $z\bar{z} = |z|^2 \Rightarrow$ Purely real
- $|z_1 z_2| = |z_1||z_2|$

  In general $|z_1 z_2 z_3 ... z_n| = |z_1||z_2||z_3|...|z_n|$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, (z_2 \neq 0)$
- $|z^n| = |z|^n, n \in N$
- $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2)$

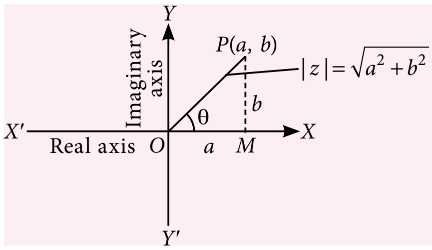
  or $|z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1\bar{z}_2)$
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Rightarrow \dfrac{z_1}{z_2}$ is purely real
- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

  (Law of parallelogram)

VARIOUS REPRESENTATIONS OF A COMPLEX NUMBER

A complex number can be represented in the following form:

- **Geometrical representation (Cartesian representation):** The complex number $z = a + ib = (a, b)$ is represented by a point P whose coordinates are referred to rectangular axes XOX' and YOY' which are called real and imaginary axis respectively. This plane is called argand plane or argand diagram or complex plane or Gaussian plane.



Angle of any complex number with positive direction of x–axis is called amplitude or argument of z.

i.e., $amp(z) = \arg(z) = \tan^{-1}\left(\dfrac{b}{a}\right)$

- **Trigonometrical (Polar) representation :** In ΔOPM, let $OP = r$, then $a = r\cos\theta$ and $b = r\sin\theta$. Hence z can be expressed as $z = r(\cos\theta + i\sin\theta)$

  where $r = |z|$ and $\theta =$ principal value of argument of z.

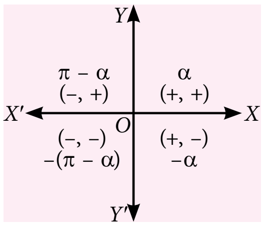
  For general values of the argument
- $z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$

  Sometimes $(\cos\theta + i\sin\theta)$ is written in short as cisθ.
- **Vector representation :** If P is the point (a, b) on the argand plane corresponding to the complex number $z = a + ib$.

Then $\overrightarrow{OP} = a\hat{i} + b\hat{j}$, $\therefore$ $|\overrightarrow{OP}| = \sqrt{a^2+b^2} = |z|$ and

$arg(z)$ = direction of the vector $\overrightarrow{OP} = \tan^{-1}\left(\frac{b}{a}\right)$

- **Eulerian representation (Exponential form) :** Since we have $e^{i\theta} = \cos\theta + i\sin\theta$ and thus z can be expressed as $z = re^{i\theta}$, where $|z| = r$ and $\theta = arg(z)$.
- **Principal value of arg (z) :** The value θ of the argument, which satisfies the inequality $-\pi < \theta \le \pi$ is called the principal value of argument, where



$\alpha = \tan^{-1}\left|\frac{b}{a}\right|$ (acute angle)

and principal values of argument z will be α, $\pi - \alpha$, $-\pi + \alpha$ and $-\alpha$ as the point z lies in the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} quadrants respectively.

- **Properties of arguments**
  - $arg(z_1z_2) = arg(z_1) + arg(z_2) + 2k\pi$, $\forall$ $K \in I$
    In general $arg(z_1z_2z_3...z_n) = arg(z_1) + arg(z_2) + arg(z_3) + ... + arg(z_n) + 2k\pi$ $\forall$ $k \in I$
  - $arg(z_1\bar{z}_2) = arg(z_1) - arg(z_2)$
  - $arg\left(\frac{z_1}{z_2}\right) = arg\, z_1 - arg\, z_2 + 2k\pi$ $\forall$ $k \in I$
  - $arg\left(\frac{z}{\bar{z}}\right) = 2arg\, z + 2k\pi$ $\forall$ $k \in I$
  - $arg(z^n) = n\, arg\, z + 2k\pi$ $\forall$ $k \in I$
  - If $arg\left(\frac{z_2}{z_1}\right) = \theta$, then $arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$, where $k \in I$
  - $arg\, \bar{z} = -arg\, z = arg\frac{1}{z}$
  - $arg(z - \bar{z}) = (4k \pm 1)\frac{\pi}{2}$, z not being purely imaginary
  - $arg(-z) = arg(z) + (2n + 1)\pi$
  - $arg(z) - \arg(\bar{z}) = \pm\pi$ (If z is purely imaginary)
  - $z_1\bar{z}_2 + \bar{z}_1z_2 = 2|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$
    where $\theta_1 = arg(z_1)$ and $\theta_2 = arg(z_2)$
  - The general value of $arg(\bar{z})$ is $2n\pi - arg(z)$.

SQUARE ROOT OF A COMPLEX NUMBER

Let $z = a + ib$ be a complex number,

$$\text{Then } \sqrt{a+ib} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right], \text{ for } b > 0$$

$$= \pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right], \text{ for } b < 0.$$

To find the square root of $a - ib$, replace i by $-i$ in the above results.

LOGARITHM OF A COMPLEX NUMBER

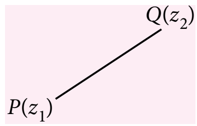
Let $\log(x + iy) = \log_e(re^{i\theta}) = \log_e r + \log_e e^{i\theta} = \log_e r + i\theta$

$$= \log_e\sqrt{(x^2+y^2)} + i\tan^{-1}\left(\frac{y}{x}\right)$$

Hence, $\log_e(z) = \log_e|z| + i\, ampz$

GEOMETRY OF COMPLEX NUMBERS

- **Distance formula :** The distance between two points $P(z_1)$ and $Q(z_2)$ is given by $PQ = |z_2 - z_1|$ $= |\text{affix of } Q - \text{affix of } P|$



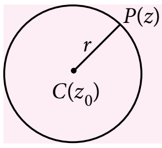
- **Section formula :** If $R(z)$ divides the line segment joining $P(z_1)$ and $Q(z_2)$ in the ratio $m_1 : m_2 (m_1, m_2 > 0)$ then
  - For internal division $z = \dfrac{m_1z_2 + m_2z_1}{m_1 + m_2}$
  - For external division $z = \dfrac{m_1z_2 - m_2z_1}{m_1 - m_2}$
- **Equation of a straight line**
  - **Parametric form :** Equation of a straight line joining the point having affixes z_1 and z_2 is $z = tz_1 + (1 - t)z_2$, when $t \in R$
  - **Non-parametric form :** Equation of a straight line joining the points having affixes z_1 and z_2 is $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$

    $\Rightarrow z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0.$
  - **Condition of Collinearity of three points :** Three points $A(z_1)$, $B(z_2)$ and $C(z_3)$ are collinear if, $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$

    or $\dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \dfrac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = \dfrac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}$
  - **General equation of a straight line :** The general equation of a straight line is of the form $\bar{a}z + a\bar{z} + b = 0$, where a is non zero complex number and b is any real number.
  - **Slope of a line :** The complex slope of the line $\bar{a}z + a\bar{z} + b = 0$ is $-\dfrac{a}{\bar{a}} = -\dfrac{\text{coeff. of } \bar{z}}{\text{coeff. of } z}$ and real slope of the line $\bar{a}z + a\bar{z} + b = 0$ is $-i\dfrac{(a+\bar{a})}{(a-\bar{a})}$.

- **Length of perpendicular :** The length of perpendicular from a point z_1 to the line $\bar{a}z + a\bar{z} + b = 0$ is given by $\dfrac{|\bar{a}z_1 + a\bar{z}_1 + b|}{|a| + |\bar{a}|}$ or $\dfrac{|\bar{a}z_1 + a\bar{z}_1 + b|}{2|a|}$

- **Equation of a circle :**
  - The equation of a circle whose centre is at point having affix z_0 and radius r is $|z - z_0| = r$

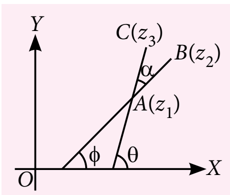


  - If the centre of the circle is at origin and radius r, then its equation is $|z| = r$.
  - $|z - z_0| < r$ represents interior of a circle, $|z - z_0| = r$ lie on the circle and $|z - z_0| > r$ represents exterior of the circle.
  - **General equation of a circle :** The general equation of the circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ (where a is complex number and $b \in R$) with centre and radius as $-a$ and $\sqrt{|a|^2 - b} = \sqrt{a\bar{a} - b}$ respectively.
  - **Equation of circle in diametric form :** If end points of diameter represented by $A(z_1)$ and $B(z_2)$ and $P(z)$ be any point on the circle then, $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$.
    which is required equation of circle in diametric form.

ROTATION THEOREM

Rotational theorem is used to find the angle between two intersecting lines. This is also known as coni method.

Let z_1, z_2 and z_3 be the affixes of three points A, B and C respectively taken on argand plane.



Then we have $\overrightarrow{AC} = z_3 - z_1$

and $\overrightarrow{AB} = z_2 - z_1$

and let $arg\ \overrightarrow{AC} = arg(z_3 - z_1) = \theta$ and $\overrightarrow{AB} = arg(z_2 - z_1) = \phi$

Let $\angle CAB = \alpha$,

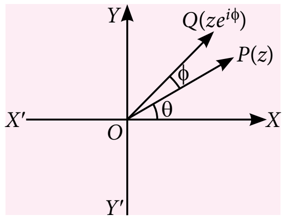
$\because\ \angle CAB = \alpha = \theta - \phi = arg\ \overrightarrow{AC} - arg\ \overrightarrow{AB}$

$$= arg(z_3 - z_1) - arg(z_2 - z_1) = arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

or angle between AC and AB

$$= arg\left(\frac{\text{affix of } C - \text{affix of } A}{\text{affix of } B - \text{affix of } A}\right)$$

- **Complex number as a rotating arrow in the argand plane :** Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ ...(i)
  $i{\cdot}e^{i\theta}$ be a complex number representing a point P in the argand plane.

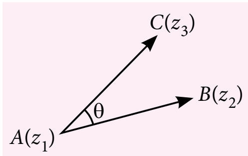


Then $OP = |z| = r$
and $\angle POX = \theta$
Now consider complex number
$z_1 = ze^{i\phi}$
or $z_1 = re^{i\theta} \cdot e^{i\phi}$
$= re^{i(\theta + \phi)}$ {From (i)}

Clearly the complex number z_1 represents a point Q in the argand plane, when $OQ = r$ and $\angle QOX = \theta + \phi$.

Clearly multiplication of z with $e^{i\phi}$ rotates the vector $\overrightarrow{OP}$ through angle ϕ in anticlockwise sense. Similarly multiplication of z with $e^{-i\phi}$ will rotate the vector $\overrightarrow{OP}$ in clockwise sense.

- If z_1, z_2 and z_3 are the affixes of the points A, B and C such that $AC = AB$ and $\angle CAB = \theta$.



  Therefore, $\overrightarrow{AB} = z_2 - z_1$, $\overrightarrow{AC} = z_3 - z_1$.
  Then $\overrightarrow{AC}$ will be obtained by rotating $\overrightarrow{AB}$ through an angle θ in anticlockwise sense, and therefore,
  $\overrightarrow{AC} = \overrightarrow{AB}e^{i\theta}$ or $(z_3 - z_1) = (z_2 - z_1)e^{i\theta}$

  or $\dfrac{z_3 - z_1}{z_2 - z_1} = e^{i\theta}$

- If A, B and C are three points in argand plane such that $AC = AB$ and $\angle CAB = \theta$ then use the rotation about A to find $e^{i\theta}$, but if $AC \neq AB$ use coni method.
- If four points z_1, z_2, z_3 and z_4 are concyclic then

  $\dfrac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_2)(z_1 - z_3)}$ is real

  or $arg\left(\dfrac{(z_2 - z_3)(z_4 - z_1)}{(z_1 - z_3)(z_4 - z_2)}\right) = \pm\pi, 0$

TRIANGLE INEQUALITIES

In any triangle, sum of any two sides is greater than the third side and difference of any two sides is less than the third side. By applying this basic concept to the set of complex numbers we are having the following results.

- $|z_1 \pm z_2| \le |z_1| + |z_2|$
- $|z_1 \pm z_2| \ge ||z_1| - |z_2||$

STANDARD LOCI IN THE ARGAND PLANE

If z is a variable point and z_1, z_2 are two fixed points in the argand plane, then

- $|z - z_1| = |z - z_2| \Rightarrow$ Locus of z is the perpendicular bisector of the line segment joining z_1 and z_2.
- $|z - z_1| + |z - z_2| = \text{constant} > |z_1 - z_2|$
  $\Rightarrow$ Locus of z is an ellipse
- $|z - z_1| + |z - z_2| = |z_1 - z_2|$
  $\Rightarrow$ Locus of z is the line segment joining z_1 and z_2.
- $|z - z_1| - |z - z_2| = |z_1 - z_2|$
  $\Rightarrow$ Locus of z is a straight line joining z_1 and z_2 but z does not lie between z_1 and z_2.
- $|z - z_1| - |z - z_2| = \text{constant} (> 0 \text{ but} \ne |z_1 - z_2|)$
  $\Rightarrow$ Locus of z is a hyperbola.
- $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2 \Rightarrow$ Locus of z is a circle with z_1 and z_2 as the extremities of diameter.
- $|z - z_1| = k|z - z_2|, (k \ne 1) \Rightarrow$ Locus of z is a circle.
- $arg\left(\dfrac{z-z_1}{z-z_2}\right) = \alpha$ (fixed) $\Rightarrow$ Locus of z is a segment of circle.
- $arg\left(\dfrac{z-z_1}{z-z_2}\right) = \pm\pi/2 \Rightarrow$ Locus of z is a circle with z_1 and z_2 as the vertices of diameter.
- $arg\left(\dfrac{z-z_1}{z-z_2}\right) = 0$ or $\pi \Rightarrow$ Locus of z is a straight line passing through z_1 and z_2.

DE' MOIVRE'S THEOREM

- If n is any rational number, then
  $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
- If $z = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$
  $(\cos\theta_3 + i\sin\theta_3)...(\cos\theta_n + i\sin\theta_n)$
  then $z = \cos(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n)$ where $\theta_1, \theta_2, \theta_3...\theta_n \in R$.
- If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then $z^{1/n} = r^{1/n}\left[\cos\left(\dfrac{2k\pi+\theta}{n}\right) + i\sin\left(\dfrac{2k\pi+\theta}{n}\right)\right]$,
  where $k = 0, 1, 2, 3,...(n - 1)$.

Deductions: If $n \in Q$, then

- $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$
- $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$
- $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$
- $(\sin\theta + i\cos\theta)^n = \cos n\left(\dfrac{\pi}{2} - \theta\right) + i\sin n\left(\dfrac{\pi}{2} - \theta\right)$

This theorem is not valid when n is not a rational number or the complex number is not in the form of $\cos\theta + i\sin\theta$.

ROOTS OF A COMPLEX NUMBER

- **n^{th} roots of complex number ($z^{1/n}$)**
  Let $z = r(\cos\theta + i\sin\theta)$ be a complex number. By using De'moivre's theorem n^{th} roots having n distinct values of such a complex number are given by
  $$z^{1/n} = r^{1/n}\left[\cos\frac{2k\pi+\theta}{n} + i\sin\frac{2k\pi+\theta}{n}\right],$$
  where $k = 0, 1, 2, ..., (n - 1)$.
- **Properties of the roots of $z^{1/n}$**
  - All roots of $z^{1/n}$ are in geometrical progression with common ratio $e^{2\pi i/n}$.
  - Sum of all roots of $z^{1/n}$ is always equal to zero.
  - Product of all roots of $z^{1/n} = (-1)^{n-1}$.
  - Modulus of all roots of $z^{1/n}$ are equal and each equal to $r^{1/n}$ or $|z|^{1/n}$.
  - Amplitude of all the roots of $z^{1/n}$ are in A.P. with common difference $\dfrac{2\pi}{n}$.
  - All roots of $z^{1/n}$ lies on the circumference of a circle whose centre is origin and radius equal to $|z|^{1/n}$. Also these roots divides the circle into n equal parts and forms a polygon of n sides.
- **The n^{th} roots of unity :** The n^{th} roots of unity are given by the solution set of the equation
  $x^n = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$
  $x = [\cos 2k\pi + i\sin 2k\pi]^{1/n}$
  $$x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \text{ where } k = 0, 1, 2, ..., (n-1).$$
- **Properties of n^{th} roots of unity**
  - Let $\alpha = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n} = e^{i(2\pi/n)}$, the n^{th} roots of unity can be expressed in the form of a series *i.e.*, $1, \alpha, \alpha^2, ... \alpha^{n-1}$. Clearly the series is G.P. with common ratio α *i.e.*, $e^{i(2\pi/n)}$.
  - The sum of all n roots of unity is zero *i.e.*, $1 + \alpha + \alpha^2 + ... + \alpha^{n-1} = 0$.
  - Product of all n roots of unity is $(-1)^{n-1}$.
  - Sum of p^{th} power of n roots of unity
    $1 + \alpha^p + \alpha^{2p} + ... + \alpha^{(n-1)p}$
    $$= \begin{cases} 0, \text{when } p \text{ is not multiple of } n \\ n, \text{when } p \text{ is a multiple of } n \end{cases}$$

- The n, n^{th} roots of unity if represented on a complex plane locate their positions at the vertices of a regular polygon of n sides inscribed in a unit circle having centre at origin, one vertex on positive real axis.

- **Cube roots of unity :** Cube roots of unity are the solution set of the equation $x^3 - 1 = 0 \Rightarrow x = (1)^{1/3}$

  $\Rightarrow \quad x = (\cos 0 + i \sin 0)^{1/3}$

  $\Rightarrow \quad x = \cos\frac{2k\pi}{3} + i\sin\left(\frac{2k\pi}{3}\right)$, where $k = 0, 1, 2$

  Therefore roots are $1, \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$,

  $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ or $1, e^{2\pi i/3}, e^{4\pi i/3}$

  Alternative : $x = (1)^{1/3} \Rightarrow x^3 - 1 = 0$

  $\Rightarrow \quad (x - 1)(x^2 + x + 1) = 0$

  $x = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

  If one of the complex root is ω, then other root will be ω^2 or vice-versa.

- **Properties of cube roots of unity**
  - $1 + \omega + \omega^2 = 0$
  - $\omega^3 = 1$
  - $1 + \omega^n + \omega^{2n} = \begin{cases} 0, \text{if } n \text{ is not a multiple of } 3 \\ 3, \text{if } n \text{ is a multiple of } 3 \end{cases}$
  - The cube roots of unity, when represented on complex plane, lie on vertices of an equilateral triangle inscribed in a unit circle having centre at origin, one vertex being on positive real axis.
  - Cube root of -1 are $-1, -\omega, -\omega^2$.

IMPORTANT POINTS

- $0 = 0 + i\cdot 0$, is the identity element for addition.
- $1 = 1 + i\cdot 0$ is the identity element for multiplication.
- The additive inverse of a complex number $z = a + ib$ is $-z$ (*i.e.* $-a - ib$).
- For every non-zero complex number z, the multiplicative inverse of z is $\frac{1}{z}$.
- $|z| \geq |\text{Re}(z)| \geq \text{Re}(z)$ and $|z| \geq |\text{Im}(z)| \geq \text{Im}(z)$
- $\frac{z}{|z|}$ is always a unimodular complex number if $z \neq 0$.
- $|\text{Re}(z)| + |\text{Im}(z)| \leq \sqrt{2}\,|z|$
- If $\left|z + \frac{1}{z}\right| = a$, the greatest and least values of $|z|$ are respectively $\frac{a+\sqrt{a^2+4}}{2}$ and $\frac{a-\sqrt{a^2+4}}{2}$.
- If $|z_1| \leq 1, |z_2| \leq 1$ then
  - $|z_1 - z_2|^2 \leq (|z_1| - |z_2|)^2 + (arg(z_1) - arg(z_2))^2$.
  - $|z_1 + z_2|^2 \geq (|z_1| + |z_2|)^2 - (arg(z_1) - arg(z_2))^2$.
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$.
- $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2| \cos(\theta_1 - \theta_2)$.
- If $|z_1| = |z_2|$ and $amp(z_1) + amp(z_2) = 0$, then z_1 and z_2 are conjugate complex numbers of each other
- The area of the triangle whose vertices are z, iz and $z + iz$ is $\frac{1}{2}|z|^2$.
- The area of the triangle with vertices z, wz and $z + wz$ is $\frac{\sqrt{3}}{4}|z^2|$.
- If z_1, z_2, z_3 be the vertices of an equilateral triangle and z_0 be the circumcentre, then $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.
- If $z_1, z_2, z_3...z_n$ be the vertices of a regular polygon of n sides and z_0 be its centroid, then $z_1^2 + z_2^2 + ... + z_n^2 = nz_0^2$.
- If z_1, z_2, z_3 be the vertices of a triangle, then the triangle is equilateral iff

  $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$

  or $\quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

  or $\quad \frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$
- If z_1, z_2, z_3 be the affixes of the vertices A, B, C respectively of a triangle ABC, then its orthocentre is $\frac{a(\sec A)z_1 + b(\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C}$
- $\text{Re}(iz) = -\text{Im}(z)$, $\text{Im}(iz) = \text{Re}(z)$.
- If the complex numbers z_1 and z_2 are such that the sum $z_1 + z_2$ is a real number, then they are not necessarily conjugate complex.
- If z_1 and z_2 are two complex numbers such that the product z_1z_2 is a real number, then they are not necessarily conjugate complex.
- If ω and ω^2 are the complex cube roots of unity, then

  $(a\omega + b\omega^2)(a\omega^2 + b\omega) = a^2 + b^2 - ab$

  $(a + b)(a\omega + b\omega)(a\omega^2 + b^2\omega) = a^3 + b^3$

  $(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) = a^2 + b^2 + c^2 - ab - bc - ca$

  $(a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) = a^3 + b^3 + c^3 - 3abc$
- If three points z_1, z_2, z_3 connected by relation $az_1 + bz_2 + cz_3 = 0$ where $a + b + c = 0$, then the three points are collinear.
- If z is a complex number, then e^z is periodic.
- If three complex numbers are in A.P., then they lie on a straight line in the complex plane.

PROBLEMS

Single Correct Answer Type

1. The value of $i^{1+3+5+...+(2n+1)}$ is
   (a) i if n is even, $-i$ if n is odd
   (b) 1 if n is even, -1 if n is odd
   (c) 1 if n is odd, -1 if n is even
   (d) i if n is even, -1 if n is odd

2. If $x+\frac{1}{x}=2\cos\theta$, then x is equal to
   (a) $\cos\theta + i\sin\theta$ (b) $\cos\theta - i\sin\theta$
   (c) $\cos\theta \pm i\sin\theta$ (d) $\sin\theta \pm i\cos\theta$

3. $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ will be real, if $\theta =$
   (a) $2n\pi$ (b) $n\pi + \frac{\pi}{2}$
   (c) $n\pi$ (d) None of these
   [where n is an integer]

4. The real part of $(1-\cos\theta + 2i\sin\theta)^{-1}$ is
   (a) $\frac{1}{3+5\cos\theta}$ (b) $\frac{1}{5-3\cos\theta}$
   (c) $\frac{1}{3-5\cos\theta}$ (d) $\frac{1}{5+3\cos\theta}$

5. If $x+iy=\frac{3}{2+\cos\theta+i\sin\theta}$, then x^2+y^2 is equal to
   (a) $3x-4$ (b) $4x-3$
   (c) $4x+3$ (d) None of these

6. If $\frac{(p+i)^2}{2p-i}=\mu+i\lambda$, then $\mu^2+\lambda^2$ is equal to
   (a) $\frac{(p^2+1)^2}{4p^2-1}$ (b) $\frac{(p^2-1)^2}{4p^2-1}$
   (c) $\frac{(p^2-1)^2}{4p^2+1}$ (d) $\frac{(p^2+1)^2}{4p^2+1}$

7. If $z(1+a)=b+ic$ and $a^2+b^2+c^2=1$, then $\frac{1+iz}{1-iz}=$
   (a) $\frac{a+ib}{1+c}$ (b) $\frac{b-ic}{1+a}$
   (c) $\frac{a+ic}{1+b}$ (d) None of these

8. If $a=\cos\theta+i\sin\theta$, then $\frac{1+a}{1-a}=$
   (a) $\cot\theta$ (b) $\cot\frac{\theta}{2}$
   (c) $i\cot\frac{\theta}{2}$ (d) $i\tan\frac{\theta}{2}$

9. The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other for
   (a) $x=n\pi$ (b) $x=\left(n+\frac{1}{2}\right)\pi$
   (c) $x=0$ (d) No value of x

10. The number of solutions of the equation $z^2+\bar{z}=0$ is
    (a) 1 (b) 2 (c) 3 (d) 4

11. If $\frac{z-i}{z+i}(z\neq -i)$ is a purely imaginary number, then $z\cdot\bar{z}$ is equal to
    (a) 0 (b) 1
    (c) 2 (d) None of these

12. The maximum value of $|z|$ where z satisfies the condition $\left|z+\frac{2}{z}\right|=2$ is
    (a) $\sqrt{3}-1$ (b) $\sqrt{3}+1$
    (c) $\sqrt{3}$ (d) $\sqrt{2}+\sqrt{3}$

13. If z_1 and z_2 be complex numbers such that $z_1\neq z_2$ and $|z_1|=|z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\frac{(z_1+z_2)}{(z_1-z_2)}$ may be
    (a) Purely imaginary (b) Real and positive
    (c) Real and negative (d) None of these

14. The product of two complex numbers each of unit modulus is also a complex number, of
    (a) Unit modulus
    (b) Less than unit modulus
    (c) Greater than unit modulus
    (d) None of these

15. If $|z_1|=|z_2|=.....=|z_n|=1$, then the value of $|z_1+z_2+z_3+....+z_n|=$
    (a) 1
    (b) $|z_1|+|z_2|+....+|z_n|$
    (c) $\left|\frac{1}{z_1}+\frac{1}{z_2}+.........+\frac{1}{z_n}\right|$
    (d) None of these

16. The values of z for which $|z+i|=|z-i|$ are
    (a) Any real number
    (b) Any complex number
    (c) Any natural number
    (d) None of these

17. Let z be a complex number (not lying on X-axis) of maximum modulus such that $\left|z+\frac{1}{z}\right|=1$. Then

(a) $\text{Im}(z) = 0$ (b) $\text{Re}(z) = 0$
(c) $\text{amp}(z) = \pi$ (d) None of these

18. The minimum value of $|2z - 1| + |3z - 2|$ is
(a) 0 (b) 1/2 (c) 1/3 (d) 2/3

19. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $arg\,(z_1) - arg\,(z_2)$ is equal to

(a) $-\pi$ (b) $-\frac{\pi}{2}$ (c) $\frac{\pi}{2}$ (d) 0

20. If $\bar{z}$ be the conjugate of the complex number z, then which of the following relations is false ?
(a) $|z| = |\bar{z}|$ (b) $z \cdot \bar{z} = |\bar{z}|^2$
(c) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$ (d) $arg\,(z) = arg\,(\bar{z})$

21. If for complex numbers z_1 and z_2, $arg(z_1/z_2) = 0$, then $|z_1 - z_2|$ is equal to
(a) $|z_1| + |z_2|$ (b) $|z_1| - |z_2|$
(c) $||z_1| - |z_2||$ (d) 0

22. $|z_1 + z_2| = |z_1| + |z_2|$ is possible if
(a) $z_2 = \bar{z}_1$ (b) $z_2 = \frac{1}{z_1}$
(c) $arg(z_1) = arg(z_2)$ (d) $|z_1| = |z_2|$

23. If $-1+\sqrt{-3} = re^{i\theta}$, then θ is equal to
(a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $-\frac{2\pi}{3}$

24. The value of $(-i)^{1/3}$ is
(a) $\frac{1+\sqrt{3}i}{2}$ (b) $\frac{1-\sqrt{3}i}{2}$
(c) $\frac{-\sqrt{3}-i}{2}$ (d) $\frac{3\sqrt{3}+i}{2}$

25. The amplitude of $e^{e^{-i\theta}}$ is equal to
(a) $\sin\theta$ (b) $-\sin\theta$ (c) $e^{\cos\theta}$ (d) $e^{\sin\theta}$

26. The real part of $(1 - i)^{-i}$ is
(a) $e^{-\pi/4}\cos\left(\frac{1}{2}\log 2\right)$
(b) $-e^{-\pi/4}\sin\left(\frac{1}{2}\log 2\right)$
(c) $e^{\pi/4}\cos\left(\frac{1}{2}\log 2\right)$ (d) $e^{-\pi/4}\sin\left(\frac{1}{2}\log 2\right)$

27. $i\log\left(\frac{x-i}{x+i}\right)$ is equal to
(a) $\pi + 2\tan^{-1} x$ (b) $\pi - 2\tan^{-1} x$
(c) $-\pi + 2\tan^{-1}x$ (d) $-\pi - 2\tan^{-1} x$

28. The equation $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, $b \in R$ represents a circle if
(a) $|a|^2 = b$ (b) $|a|^2 > b$
(c) $|a|^2 < b$ (d) None of these

29. Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle, then $z_1^2 + z_2^2 + z_3^2 =$
(a) z_0^2 (b) $-z_0^2$ (c) $3z_0^2$ (d) $-3z_0^2$

30. If the vertices of a quadrilateral be $A = 1 + 2i$, $B = -3 + i$, $C = -2 - 3i$ and $D = 2 - 2i$, then the quadrilateral is
(a) Parallelogram (b) Rectangle
(c) Square (d) Rhombus

31. If ω is a complex number satisfying $\left|\omega+\frac{1}{\omega}\right|=2$, then maximum distance of ω from origin is
(a) $2+\sqrt{3}$ (b) $1+\sqrt{2}$
(c) $1+\sqrt{3}$ (d) None of these

Multiple Correct Answer Type

32. Let $z_1, z_2, z_3, ..., z_n$ are the complex numbers such that $|z_1| = |z_2| = ... = |z_n| = 1$. If $z = \left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right)$ then
(a) z is purely imaginary
(b) z is real
(c) $0 < z \le n^2$
(d) z is a complex number of the form $a + ib$

33. Let x_1, x_2 are the roots of the quadratic equation $x^2 + ax + b = 0$ where a,b are complex numbers and y_1, y_2 are the roots of the quadratic equation $y^2 + |a|y + |b| = 0$. If $|x_1| = |x_2| = 1$ then
(a) $|y_1| = 1$ (b) $|y_2| = 1$
(c) $|y_1| \neq |y_2|$ (d) $|y_1| = |y_2| = 2$

34. If the equation $z^3 + (3 + i)z^2 - 3z - (m + i) = 0$ where $m \in R$, has atleast one real root then 'm' can have the value equal to
(a) 1 (b) 2 (c) 3 (d) 5

35. Let z_1, z_2, z_3 in G.P. be roots of the equation $z^3 - bz^2 + 3z - 1 = 0$ then
(a) $z_2 = 1$ (b) $z_2 = 2$
(c) $b = 3$ (d) b can be -3

36. The complex numbers satisfying the equation $(3z + 1)(4z + 1)(6z+ 1)(12z + 1) = 2$ is /are

(a) $\dfrac{\sqrt{33}-5}{24}$ (b) $\dfrac{\sqrt{33}+5}{24}$

(c) $\dfrac{-i\sqrt{23}-5}{24}$ (d) $\dfrac{-i\sqrt{23}+5}{24}$

37. If all the three roots of $az^3 + bz^2 + cz + d = 0$ have negative real parts ($a, b, c \in R$) then

(a) $ab > 0$ (b) $bc > 0$

(c) $ad > 0$ (d) $bc - ad > 0$

Comprehension Type

Paragraph for Q. No. 38 to 40

The complex slope M of a line joining two points z_1 and z_2 in complex plane is defined as $M = \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$ .Its real slope m is $\tan\theta$, where θ is inclination of the line .

38. $M =$

(a) $\dfrac{1+im}{1-im}$

(b) $\dfrac{2m}{1+m^2}+i\left(\dfrac{1-m^2}{1+m^2}\right)$

(c) $\dfrac{2m}{1+m^2}+i\left(\dfrac{1+m^2}{1-m^2}\right)$

(d) $\dfrac{1-im}{m-i}$

39. The inclination of a line whose complex slope is $-\omega^2$ (where ω is a non real cube root of unity) is

(a) $\dfrac{2\pi}{3}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{6}$ (d) $\dfrac{\pi}{12}$

40. Which of the following is false?

(a) $|M| = 1$

(b) $M = m$ is never possible

(c) $m = i\left(\dfrac{M-1}{M+1}\right)$

(d) $M = cis\, 2\theta$

Paragraph for Q. No. 41 to 43

If α is any of 7$^{\text{th}}$ roots of unity, then $\alpha = \dfrac{\text{cis}\,2K\pi}{7}$ (K = 0 to 6) and $\sum_{i=1}^{6}\alpha^i = -1(\alpha \neq 1)$ and $\alpha^7 = 1$

41. The equation whose roots are $\alpha + \alpha^2 + \alpha^4$ and $\alpha^3 + \alpha^5 + \alpha^6$ is $(\alpha \neq 1)$

(a) $x^2 + x - 2 = 0$ (b) $x^2 + x + 2 = 0$

(c) $x^2 - x + 2 = 0$ (d) $x^2 - x - 2 = 0$

42. If $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6$ then for $\alpha \neq 1, f(x)+f(\alpha x)+f(\alpha^2 x)+f(\alpha^3 x)+....+f(\alpha^6 x) =$

(a) 42 (b) 21 (c) 14 (d) 7

43. $\sin\dfrac{\pi}{7}\sin\dfrac{2\pi}{7}\sin\dfrac{3\pi}{7} = \sin\dfrac{4\pi}{7}\sin\dfrac{5\pi}{7}\sin\dfrac{6\pi}{7} =$

(a) $\dfrac{\sqrt{7}}{16}$ (b) $\dfrac{\sqrt{7}}{8}$ (c) $\dfrac{\sqrt{7}}{32}$ (d) $\dfrac{\sqrt{7}}{64}$

Matrix-Match Type

44. Match the following.

	Column-I		Column-II
(A)	The maximum value of $\lvert\lvert z-\omega\rvert - \lvert z-\bar{\omega}\rvert\rvert$ (where $\lvert z\rvert = 5$ and $\omega, \bar{\omega}$ complex cube root of unity) is	(p)	0
(B)	Tangent drawn to circle $(x-1)^2 + (y-1)^2 = 5$ at a point P meets the line $2x + y + 6 = 0$ at Q on the x-axis then the value of $\dfrac{(PQ)^2}{2}$ is	(q)	2/3
(C)	One vertex of an equilateral triangle is at the origin and the other two vertices are given by $2z^2 + 2z + k = 0$, then k is	(r)	$\sqrt{3}$
		(s)	6

45. Match the following.

	Column –I		Column –II
(A)	If $z = \dfrac{z_1 + i\,\bar{z}_2}{z_2 + i\,\bar{z}_1}$ then $\lvert z\rvert$ equals	(p)	0
(B)	If $z = \dfrac{z-2i}{z+2i}$ be purely imaginary then $\lvert z\rvert$ equals	(q)	1
(C)	If $\lvert z+6\rvert = \lvert 2z+3\rvert$ then $\lvert z\rvert$ equals	(r)	2
(D)	Let $\omega \neq 1$ be a cube root of unity and $z = \dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} + \dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}$ then $\lvert z\rvert$ equals	(s)	3
		(t)	4

Integer Answer Type

46. Let λ, z_0 be two complex numbers. $A(z_1)$, $B(z_2)$, $C(z_3)$ be the vertices of a triangle such that $z_1 = z_0 + \lambda$, $z_2 = z_0 + \lambda e^{i\pi/4}$, $z_3 = z_0 + \lambda e^{i7\pi/11}$ and $\angle ABC = \frac{3k\pi}{22}$ then the value of k is

47. If the argument of $(z - a)(\bar{z} - b)$ is equal to that of $\frac{(\sqrt{3}+i)(1+\sqrt{3}\,i)}{1+i}$, where a, b are real numbers. If locus of z is a circle with centre $\frac{3+i}{2}$ then find $a + b$.

48. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are the roots of the equation $x^5 - 1 = 0$, where $\alpha_k = \alpha^{k-1}$, $\alpha = e^{i2\pi/5}$ and $\lambda = \alpha_3^{1001}$, $\mu = \alpha_4^{(669+1/3)}$, $\nu = \alpha_5^{(503+1/2)}$, then $[|\lambda^{2011} + \mu^{2011} + \nu^{2011}|]$ (where $[\cdot]$ denotes the greatest integer function) is

49. Two lines $zi - \bar{z}i + 2 = 0$ and $z(1 + i) + \bar{z}(1 - i) + 2 = 0$ intersect at a point P. There is a complex number $\alpha = x + iy$ at a distance of 2 units from the point P which lies on line $z(1 + i) + \bar{z}(1 - i) + 2 = 0$. Find $[|x|]$ (where $[\cdot]$ represents greatest integer function).

50. Suppose that w is the imaginary $(2009)^{\text{th}}$ roots of unity. If $(2^{2009} - 1)\sum_{r=1}^{2008}\frac{1}{2-w^r} = (a)(2^b) + c$ where $a, b, c \in N$ and the least value of $(a + b + c)$ is $(2008)K$. The numerical value of K is

SOLUTIONS

1. (c) : Let $z = i^{[1+3+5+....+(2n+1)]}$

Clearly series is A.P. with common difference = 2

$\because\ T_n = 2n - 1$ and $T_{n+1} = 2n+1$

So, number of terms in A.P. $= n + 1$

Now, $S_{n+1} = \frac{n+1}{2}[2\cdot 1 + (n+1-1)2]$

$\Rightarrow\ S_{n+1} = \frac{n+1}{2}[2+2n] = (n+1)^2$ *i.e.* $z = i^{(n+1)^2}$

Now put $n = 1, 2, 3, 4, 5, ...$

$n = 1, z = i^4 = 1, n = 2, z = i^6 = -1,$

$n = 3, z = i^8 = 1, n = 4, z = i^{10} = -1,$

$n = 5, z = i^{12} = 1, ...$

2. (c) : $x + \frac{1}{x} = 2\cos\theta \Rightarrow x^2 - 2x\cos\theta + 1 = 0$

$\Rightarrow\ x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \Rightarrow x = \cos\theta \pm i\sin\theta$

3. (c) : We have, $\frac{3+2i\sin\theta}{1-2i\sin\theta} = \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$

$= \left(\frac{3-4\sin^2\theta}{1+4\sin^2\theta}\right) + i\left(\frac{8\sin\theta}{1+4\sin^2\theta}\right)$

Now, since it is real, therefore Im $(z) = 0$

$\Rightarrow\ \frac{8\sin\theta}{1+4\sin^2\theta} = 0 \Rightarrow \sin\theta = 0\ \therefore\ \theta = n\pi$

where $n = 0, 1, 2, 3, ...$

Remark : Check for (a), if $n = 0$, $\theta = 0$ the given number is absolutely real but (c) also satisfies this condition and in (a) and (c), (c) is most general value of θ.

4. (d) : Let $z = 0\ \{(1 - \cos\theta) + i\cdot 2\sin\theta\}^{-1}$

$= \left(2\sin\frac{\theta}{2}\right)^{-1}\left\{\sin\frac{\theta}{2} + i\cdot 2\cos\frac{\theta}{2}\right\}^{-1}$

$= \left(2\sin\frac{\theta}{2}\right)^{-1}\frac{1}{\sin\frac{\theta}{2} + i\cdot 2\cos\frac{\theta}{2}} \times \frac{\sin\frac{\theta}{2} - i\cdot 2\cos\frac{\theta}{2}}{\sin\frac{\theta}{2} - i\cdot 2\cos\frac{\theta}{2}}$

$= \frac{\sin\frac{\theta}{2} - i\cdot 2\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(\sin^2\frac{\theta}{2} + 4\cos^2\frac{\theta}{2}\right)}$

$\therefore\ \text{Re}(z) = \frac{\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(1+3\cos^2\frac{\theta}{2}\right)}$

5. (b) : If $x + iy = \frac{3}{2+\cos\theta + i\sin\theta}$

$= \frac{3(2+\cos\theta - i\sin\theta)}{(2+\cos\theta)^2 + \sin^2\theta} = \frac{6+3\cos\theta - 3i\sin\theta}{4+\cos^2\theta + 4\cos\theta + \sin^2\theta}$

$= \left[\frac{6+3\cos\theta}{5+4\cos\theta}\right] + i\left[\frac{-3\sin\theta}{5+4\cos\theta}\right]$

$x = \frac{3(2+\cos\theta)}{5+4\cos\theta}, y = \frac{-3\sin\theta}{5+4\cos\theta}$

$\therefore\ x^2 + y^2 = \frac{9}{(5+4\cos\theta)^2}[4 + \cos^2\theta + 4\cos\theta + \sin^2\theta]$

$= \frac{9}{5+4\cos\theta} = 4\left[\frac{6+3\cos\theta}{5+4\cos\theta}\right] - 3 = 4x - 3$

6. (d) : We have,

$\mu + i\lambda = \frac{(p+i)^2}{2p-i} = \frac{(p^2-1+2pi)(2p+i)}{(2p-i)(2p+i)}$

$$= \frac{2p(p^2-2)+i(5p^2-1)}{4p^2+1}$$

$$\therefore \quad \mu^2+\lambda^2 = \frac{4p^2(p^2-2)^2+(5p^2-1)^2}{(4p^2+1)^2}$$

$$= \frac{4p^6+6p^2+9p^4+1}{(4p^2+1)^2}$$

$$= \frac{p^4(4p^2+1)+2p^2(4p^2+1)+(4p^2+1)}{(4p^2+1)^2}$$

$$= \frac{p^4+2p^2+1}{4p^2+1} = \frac{(p^2+1)^2}{4p^2+1}$$

7. (a) : Given, $z(1 + a) = b + ic \Rightarrow z = \frac{b+ic}{1+a}$

$$\frac{1+iz}{1-iz} = \frac{1+i(b+ic)/(1+a)}{1-i(b+ic)/(1+a)} = \frac{1+a-c+ib}{1+a+c-ib}$$

$$= \frac{(1+a-c+ib)(1+a+c+ib)}{(1+a+c)^2+b^2}$$

$$= \frac{1+2a+a^2-b^2-c^2+2ib+2iab}{1+a^2+c^2+b^2+2ac+2(a+c)}$$

$$= \frac{a^2+b^2+c^2+2a+a^2-b^2-c^2+2ib(1+a)}{1+1+2ac+2(a+c)}$$

$$= \frac{2a(a+1)+2ib(1+a)}{2(1+a)(1+c)} = \frac{a+ib}{1+c}$$

8. (c) : $a = \cos\theta + i\sin\theta$

$$\therefore \quad \frac{1+a}{1-a} = \frac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta}.$$

On Rationalizing the denominator, we get

$$\frac{1+a}{1-a} = \frac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta} \times \frac{(1-\cos\theta)+i\sin\theta}{(1-\cos\theta)+i\sin\theta}$$

$$= \frac{(1+\cos\theta)(1-\cos\theta)+(1+\cos\theta)i\sin\theta + (1-\cos\theta)i\sin\theta + i^2\sin^2\theta}{(1-\cos\theta)^2-(i\sin\theta)^2}$$

$$= \frac{1-\cos^2\theta+i\sin\theta+i\sin\theta\cos\theta + i\sin\theta - i\sin\theta\cos\theta - \sin^2\theta}{1+\cos^2\theta-2\cos\theta+\sin^2\theta}$$

$$= \frac{1-(\cos^2\theta+\sin^2\theta)+2i\sin\theta}{1+(\cos^2\theta+\sin^2\theta)-2\cos\theta} = \frac{2i\sin\theta}{2(1-\cos\theta)}$$

$$= \frac{i\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{i\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = i\cot\frac{\theta}{2}$$

9. (d) : $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other if $\sin x = \cos x$ and $\cos 2x = \sin 2x$

or $\tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, ...$ ...(i)

and $\tan 2x = 1 \Rightarrow 2x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, ...$

$\Rightarrow x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}$ ....(ii)

There exists no value of x common in (i) and (ii). Therefore there is no value of x for which the given complex numbers are conjugate.

10. (d) : Let $z = x + iy$, so, $\bar{z} = x - iy$,

$\therefore \quad z^2+\bar{z}=0 \Leftrightarrow (x^2-y^2+x)+i(2xy-y)=0$

Equating real and imaginary parts, we get

$x^2 - y^2 + x = 0$ ...(i)

and $2xy - y = 0 \Rightarrow y = 0$ or $x = \frac{1}{2}$

If $y = 0$, then (i) gives $x^2 + x = 0 \Rightarrow x = 0$ or $x = -1$

If $x = \frac{1}{2}$, then $x^2 - y^2 + x = 0 \Rightarrow y^2 = \frac{1}{4}+\frac{1}{2} = \frac{3}{4}$

$\Rightarrow \quad y = \pm\frac{\sqrt{3}}{2}$

Hence, there are four solutions in all.

11. (b) : Here, $\frac{z-i}{z+i} = \frac{x+i(y-1)}{x+i(y+1)} \cdot \frac{x-i(y+1)}{x-i(y+1)}$

$$= \frac{(x^2+y^2-1)+i(-2x)}{x^2+(y+1)^2}$$

As $\frac{z-i}{z+i}$ is purely imaginary, we get

$x^2 + y^2 - 1 = 0 \Rightarrow x^2 + y^2 = 1 \Rightarrow z\bar{z} = 1$

12. (b) : $\left|z+\frac{2}{z}\right| = 2 \Rightarrow |z| - \frac{2}{|z|} \le 2 \Rightarrow |z|^2 - 2|z| - 2 \le 0$

$|z| \le \frac{2 \pm \sqrt{4+8}}{2} \le 1 \pm \sqrt{3}$.

Hence max. value of $|z|$ is $1+\sqrt{3}$

13. (a) : Let $z_1 = a + ib = (a, b)$ and $z_2 = c - id = (c, -d)$ where $a > 0$ and $d > 0$ ...(i)

Then $|z_1| = |z_2| \Rightarrow a^2 + b^2 = c^2 + d^2$

Now $\frac{z_1+z_2}{z_1-z_2} = \frac{(a+ib)+(c-id)}{(a+ib)-(c-id)}$

$$= \frac{[(a+c)+i(b-d)][(a-c)-i(b+d)]}{[(a-c)+i(b+d)][(a-c)-i(b+d)]}$$

$$= \frac{(a^2+b^2)-(c^2+d^2)-2(ad+bc)i}{a^2+c^2-2ac+b^2+d^2+2bd}$$

$$= \frac{-(ad+bc)i}{a^2+b^2-ac+bd} \quad \text{[using (i)]}$$

$\therefore \ \dfrac{(z_1+z_2)}{(z_1-z_2)}$ is purely imaginary.

However if $ad + bc = 0$, then $\dfrac{(z_1+z_2)}{(z_1-z_2)}$ will be equal to zero. According to the conditions of the equation, we can have $ad + bc = 0$

Remark : Assume any two complex numbers satisfying both conditions *i.e.*, $z_1 \neq z_2$ and $|z_1| = |z_2|$

Let $z_1 = 2 + i$, $z_2 = 1 - 2i$, $\therefore \ \dfrac{z_1+z_2}{z_1-z_2} = \dfrac{3-i}{1+3i} = -i$

Hence the result.

14. (a)

15. (c) : We have $|z_k| = 1$, $k = 1, 2, ... n$

$$\Rightarrow |z_k|^2 = 1 \Rightarrow \overline{z_k z k} = 1 \Rightarrow \overline{z_k} = \frac{1}{z_k}$$

Therefore, $|z_1+z_2+....+z_n| = |\overline{z_1+z_2+....+z_n}|$

$$(\because |z| = |\bar{z}|)$$

$$\Rightarrow |\bar{z}_1+\bar{z}_2+.....+\bar{z}_n| = \left|\frac{1}{z_1}+\frac{1}{z_2}+....+\frac{1}{z_n}\right|$$

16. (a) : Let, $z = x + iy$ ...(i)

Given $|z + i| = |z - i|$ or $|x + iy + i| = |x + iy - i|$

or $|x + i(y + 1)| = |x + i(y - 1)|$

$\Rightarrow \sqrt{x^2+(y+1)^2} = \sqrt{x^2+(y-1)^2}$

$\Rightarrow x^2 + (y + 1)^2 = x^2 + (y - 1)^2$

$\Rightarrow y^2 + 2y + 1 = y^2 - 2y + 1 \Rightarrow 4y = 0$ or $y = 0$

Hence from (i), we get $z = x$, where x is any real number.

17. (b) : Let $z = r(\cos\theta + i\sin\theta)$

Then $\left|z+\dfrac{1}{z}\right| = 1 \Rightarrow \left|z+\dfrac{1}{z}\right|^2 = 1$

$$\Rightarrow \left|r(\cos\theta+i\sin\theta)+\frac{1}{r}(\cos\theta-i\sin\theta)\right|^2 = 1$$

$$\Rightarrow \left(r+\frac{1}{r}\right)^2\cos^2\theta+\left(r-\frac{1}{r}\right)^2\sin^2\theta = 1$$

$$\Rightarrow r^2+\frac{1}{r^2}+2\cos 2\theta = 1 \quad \text{...(i)}$$

Since $|z| = r$ is maximum, therefore $\dfrac{dr}{d\theta} = 0$

Differentiating (i) w.r.t. θ, we get

$$2r\frac{dr}{d\theta}-\frac{2}{r^3}\frac{dr}{d\theta}-4\sin 2\theta = 0$$

Putting $\dfrac{dr}{d\theta} = 0$, we get $\sin 2\theta = 0 \Rightarrow \theta = 0$ or $\dfrac{\pi}{2}$

$\therefore$ z is purely imaginary. $(\because \theta \neq 0)$

18. (c) : Given expression, $|2z - 1| + |3z - 2|$, minimum value of $|2z - 1|$ is 0 at $z = \dfrac{1}{2}$. So value of given expression $= 0 + \dfrac{1}{2} = \dfrac{1}{2}$, minimum value of $|3z - 2|$ is 0 at $z = \dfrac{2}{3}$. So value of given expression $= \dfrac{1}{3} + 0 = \dfrac{1}{3}$.

So minimum value of given expression is $\dfrac{1}{3}$.

19. (d) : Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore \ |z_1+z_2| = [(r_1\cos\theta_1+r_2\cos\theta_2)^2+(r_1\sin\theta_1+r_2\sin\theta_2)^2]^{1/2}$

$= [r_1^2+r_1^2+2r_1r_2\cos(\theta_1-\theta_2)]^{1/2} = [(r_1+r_2)^2]^{1/2}$

$(\because |z_1 + z_2| = |z_1| + |z_2|)$

Therefore $\cos(\theta_1 - \theta_2) = 1 \Rightarrow \theta_1 - \theta_2 = 0 \Rightarrow \theta_1 = \theta_2$

Thus $arg(z_1) - arg(z_2) = 0$

20. (d) : Let $z = x + iy$, $\bar{z} = x - iy$

Since $arg(z) = \theta = \tan^{-1}\dfrac{y}{x}$

$arg(\bar{z}) = \theta = \tan^{-1}\left(\dfrac{-y}{x}\right)$

Thus $arg(z) \neq arg(\bar{z})$

21. (c) : We have, $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

where $\theta_1 = arg(z_1)$ and $\theta_2 = arg(z_2)$

Since $arg(z_1) - arg(z_2) = 0$

$\therefore \ |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2| = (|z_1| - |z_2|)^2$

$\Rightarrow |z_1 - z_2| = ||z_1| - |z_2||$

22. (c) : $|z_1 + z_2| = |z_1| + |z_2|$

$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow |z_1|^2 + |z_2|^2 + 2\text{Re}\,|z_1\bar{z}_2| = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow 2\text{Re}\,|z_1\bar{z}_2| = 2|z_1||z_2|$

$\Rightarrow 2|z_1||\bar{z}_2|\cos(\theta_1 - \theta_2) = 2|z_1||z_2|$

$\Rightarrow \cos(\theta_1 - \theta_2) = 1$ or $\theta_1 - \theta_2 = 0$

$\therefore \ arg(z_1) = arg(z_2)$

23. (c) : Here, $-1+\sqrt{-3}=re^{i\theta}$

$\Rightarrow \quad -1+i\sqrt{3}=re^{i\theta}=r\cos\theta+ir\sin\theta$

Equating real and imaginary parts, we get

$r\cos\theta=-1$ and $r\sin\theta=\sqrt{3}$

Hence, $\tan\theta=-\sqrt{3}\Rightarrow\tan\theta=\tan\frac{2\pi}{3}$. Hence $\theta=\frac{2\pi}{3}$.

24. (c) : Since $\frac{-\sqrt{3}-i}{2}=-\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$\Rightarrow \quad \left(\frac{-\sqrt{3}-i}{2}\right)^3=-\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^3=-i$

25. (b) : Let $z=e^{e^{-i\theta}}=e^{\cos\theta-i\sin\theta}=e^{\cos\theta}\,e^{-i\sin\theta}$

$z=e^{\cos\theta}[\cos(\sin\theta)-i\sin(\sin\theta)]$

$z=e^{\cos\theta}\cos(\sin\theta)-ie^{\cos\theta}\sin(\sin\theta)$

$$amp(z)=\tan^{-1}\left[-\frac{e^{\cos\theta}\sin(\sin\theta)}{e^{\cos\theta}\cos(\sin\theta)}\right]$$

$=\tan^{-1}[\tan(-\sin\theta)]=-\sin\theta$

26. (a) : Let $z=(1-i)^{-i}$. Taking log on both sides,

$\Rightarrow \quad \log z=-i\log(1-i)=-i\log\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$

$=-i\log\left(\sqrt{2}e^{-i\pi/4}\right)=-i\left[\frac{1}{2}\log 2+\log e^{-i\pi/4}\right]$

$=-i\left[\frac{1}{2}\log 2-\frac{i\pi}{4}\right]=-\frac{i}{2}\log 2-\frac{\pi}{4}$

$\Rightarrow \quad z=e^{-\pi/4}\,e^{\frac{-i}{2}(\log 2)}$

Taking real part only, we get

$$\mathrm{Re}(z)=e^{-\pi/4}\cos\left(\frac{1}{2}\log 2\right)$$

27. (b) : Let $z=i\log\left(\frac{x-i}{x+i}\right)\Rightarrow\frac{z}{i}=\log\left(\frac{x-i}{x+i}\right)$

$\Rightarrow \quad \frac{z}{i}=\log\left[\frac{x-i}{x+i}\times\frac{x-i}{x-i}\right]=\log\left[\frac{x^2-1-2ix}{x^2+1}\right]$

$\Rightarrow \quad \frac{z}{i}=\log\left[\frac{x^2-1}{x^2+1}-i\frac{2x}{x^2+1}\right]$

$\Rightarrow \quad \frac{z}{i}=\log\sqrt{\left(\frac{x^2-1}{x^2+1}\right)^2+\left(\frac{-2x}{x^2+1}\right)^2}+i\tan^{-1}\left(\frac{-2x}{x^2-1}\right)$

$[\because \log(a+ib)=\log(re^{i\theta})=\log r+i\theta$

$\qquad =\log\sqrt{a^2+b^2}+i\tan^{-1}(b/a)]$

$\Rightarrow \quad \frac{z}{i}=\log\frac{\sqrt{x^4+1-2x^2+4x^2}}{(x^2+1)}+i\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$\qquad =\log 1+i(2\tan^{-1}x)$

$\Rightarrow \quad z=i^2\,2\tan^{-1}x=-2\tan^{-1}x=\pi-2\tan^{-1}x$.

28. (b) : By adding $a\bar{a}$ on both the sides of

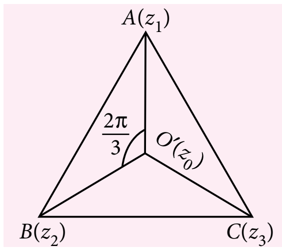
$z\bar{z}+a\bar{z}+\bar{a}z=-b$

we get, $(z+a)(\bar{z}+\bar{a})=a\bar{a}-b$

$\Rightarrow \quad |z+a|^2=|a|^2-b, \qquad \{\because z\bar{z}=|z|^2\}$

This equation will represent a circle with centre $z=-a$, if $|a|^2-b>0$, *i.e.* $|a|^2>b$ since $|a|^2=b$ represents point circle only.

29. (c) : Let r be the circumradius of the equilateral triangle and ω the cube root of unity.



Let ABC be the equilateral triangle with z_1, z_2 and z_3 as its vertices A, B and C respectively with circumcentre $O'(z_0)$.

Then the vectors $\overrightarrow{O'A}=z_1-z_0=re^{i\theta}$

$\overrightarrow{O'B}=z_2-z_0=re^{i\left(\theta+\frac{2\pi}{3}\right)}=r\omega e^{i\theta}$

$\overrightarrow{O'C}=z_3-z_0=re^{i\left(\theta+\frac{4\pi}{3}\right)}=r\omega^2e^{i\theta}$

$\therefore \quad z_1=z_0+re^{i\theta}, z_2=z_0+r\omega e^{i\theta}, z_3=z_0+r\omega^2e^{i\theta}$

Squaring and adding

$z_1^2+z_2^2+z_3^2=3z_0^2+2(1+\omega+\omega^2)z_0re^{i\theta}$

$\qquad +(1+\omega^2+\omega^4)r^2e^{i2\theta}$

$=3z_0^2$, since $1+\omega+\omega^2=0=1+\omega^2+\omega^4$

30. (c) : Given the vertices of quadrilateral $A(1+2i)$, $B(-3+i)$, $C(-2-3i)$ and $D(2-2i)$

Now, $AB=\sqrt{16+1}=\sqrt{17}, BC=\sqrt{16+1}=\sqrt{17}$

$CD=\sqrt{16+1}=\sqrt{17}, DA=\sqrt{16+1}=\sqrt{17}$

$AC=\sqrt{9+25}=\sqrt{34}, BD=\sqrt{25+9}=\sqrt{34}$

Hence it is a square.

31. (b) : Since maximum distance of any complex number ω from origin is given by $|\omega|$

therefore, $|\omega| = \left|\omega + \frac{1}{\omega} - \frac{1}{\omega}\right| \le \left|\omega + \frac{1}{\omega}\right| + \left|\frac{1}{\omega}\right| = 2 + \frac{1}{|\omega|}$

$\Rightarrow$ $|\omega|^2 - 2|\omega| - 1 \le 0 \Rightarrow |\omega| \le \frac{2 \pm 2\sqrt{2}}{2}$

Hence, max $|\omega|$ is $1+\sqrt{2}$

32. (b, c) : We have,

$$z = (z_1 + z_2 + ... + z_n)\left(\frac{1}{z_1} + \frac{1}{z_2} + ... + \frac{1}{z_n}\right)$$

$= |z_1 + z_2 + ... + z_n|^2 \le |z_1|^2 + |z_2|^2 + |z_3|^2 + ... + |z_n|^2 = n^2$

33. (a, b) : $x^2 + ax + b = 0 \Rightarrow |a| \le 2$ and $|b| = 1$

$$y = \frac{-|a| \pm \sqrt{|a|^2 - 4|b|}}{2} = \frac{-|a| \pm i\sqrt{4 - |a|^2}}{2}$$

$\therefore$ $|y| = 1$

34. (a, d) : Let α is a real root then

$\alpha^3 + (3+i)\alpha^2 - 3\alpha = m + i$

$\Rightarrow$ $\alpha^3 + 3\alpha^2 - 3\alpha - m = 0$ or $\alpha^2 - 1 = 0$

$\Rightarrow$ $\alpha = 1$ or -1 $\Rightarrow$ $m = 1$ or 5

35. (a, c) : We have z_1, z_2, z_3 in G.P.

$\therefore$ $z_2^2 = z_1z_3 \Rightarrow z_2^3 = 1$

So, $z_2 = 1, \omega, \omega^2$

Hence, $1 - b + 3 - 1 = 0 \Rightarrow b = 3$

36. (a, c) : $(3z + 1)(4z + 1)(6z + 1)(12z + 1) = 2$

$\Rightarrow$ $8(3z + 1)\ 6\ (4z + 1)\ 4\ (6z + 1)\ 2\ (12z + 1)$

$= 2 \times 8 \times 6 \times 4 \times 2$

$(24z + 8)(24z + 6)(24z + 4)(24z + 2) = 768$

Let $24z + 5 = U$

$(U + 3)(U + 1)(U - 1)(U - 3) = 768$

$\Rightarrow$ $(U^2 - 9)(U^2 - 1) = 768$

$\Rightarrow$ $U^4 - 10U^2 - 759 = 0 \Rightarrow U^2 = 33$ or -23

$\Rightarrow$ $24z + 5 = \pm\sqrt{33}$ or $\pm i\sqrt{23}$

$$z = \frac{\pm\sqrt{33} - 5}{24} \text{ or } \frac{\pm i\sqrt{23} - 5}{24}$$

37. (a, b, c, d) : Let $z_1 = x_1$; $z_2, z_3 = x_2 \pm iy_2$

$\Rightarrow$ $z_1 + z_2 + z_3 = -\frac{b}{a}$

$\Rightarrow$ $x_1 + 2x_2 = -\frac{b}{a} < 0 \Rightarrow ab > 0$

Also, $z_1 z_2 z_3 = x_1\left[x_2^2 + y_2^2\right] = -\frac{d}{a} \Rightarrow ad > 0$

Also $-\frac{bc}{a^2} < x_1\left(x_2^2 + y_2^2\right)$ $\Rightarrow$ $bc > ad$

38. (c)

39. (c)

40. (c) : Given $M = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \frac{(x_1 - x_2) + i(y_1 - y_2)}{(x_1 - x_2) - i(y_1 - y_2)}$

$$= \frac{1 + i\tan\theta}{1 - i\tan\theta} = \frac{1 + im}{1 - im}$$

$$M = i\left(\frac{2m}{1+m^2}\right) + \left(\frac{1-m^2}{1+m^2}\right)$$

$\therefore$ $M = \cos 2\theta + i \sin 2\theta = \text{cis } 2\theta \Rightarrow |M| = 1$

Also, $-\omega^2 = \frac{1 + i\sqrt{3}}{2} = \text{cis } 2\theta \Rightarrow \theta = \frac{\pi}{6}$

41. (b)

42. (d)

43. (b) : Let $\alpha = \text{cis } \frac{2\pi}{7}$ then $a = \alpha + \alpha^2 + \alpha^4$, $b = \alpha^3 + \alpha^5 + \alpha^6$

$\Rightarrow$ $a + b = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = -1$

and $ab = (\alpha + \alpha^2 + \alpha^4)(\alpha^3 + \alpha^5 + \alpha^6) = 2$

$\therefore$ Equation whose roots are a, b is $x^2 - x(-1) + 2 = 0$

$f(x) + f(\alpha x) + f(\alpha^2 x) + f(\alpha^3 x) + ... + f(\alpha^6 x)$

$= (1 + 1 + 1 + ... + 1) + 2x(1 + \alpha + \alpha^2 + \alpha^3 + ... + \alpha^6) + 3x^2(1 + \alpha^2 + \alpha^4 + ... + \alpha^{12}) + ... + 7x^6(1 + \alpha^6 + \alpha^{12} + \alpha^{18} + ... + \alpha^{36}) = 7$

Also, $\frac{x^7 - 1}{x - 1} = (x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4)(x - \alpha^5)(x - \alpha^6)$

Putting $\alpha = \text{cis } \frac{2\pi}{7}$ of applying $x \to 1$ gives

$$7 = \left|\left(1 - \text{cis}\frac{2\pi}{7}\right)\left(1 - \text{cis}\frac{4\pi}{7}\right)....\left(1 - \text{cis}\frac{12\pi}{7}\right)\right|$$

$$7 = \left(2\sin\frac{\pi}{7}\right)\left(2\sin\frac{2\pi}{7}\right)....\left(2\sin\frac{6\pi}{7}\right)$$

$$\Rightarrow \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \sin\frac{4\pi}{7}\sin\frac{5\pi}{7}\sin\frac{6\pi}{7} = \sqrt{\frac{7}{64}} = \frac{\sqrt{7}}{8}$$

44. (A) $\to$ (s), (B) $\to$ (q), (C) $\to$ (r)

(A) $\left||z - \omega| - |z - \bar{\omega}|\right|_{max} = |\omega - \bar{\omega}| = \sqrt{3}$

(B) $Q \equiv (-3, 0)$ $\therefore$ $PQ = \sqrt{S_1} = \sqrt{12}$

(C) $2z^2 + 2z + k = 0$

$\therefore$ $z = \frac{-2 \pm \sqrt{4 - 8k}}{4}$

Since 'z' is a complex number $4 - 8k$ will be negative

$\Rightarrow$ $k > \frac{1}{2}$

$\therefore$ Points are (0, 0), $\left(\frac{-1}{2}, \frac{\sqrt{2k-1}}{2}\right)$ $\left(\frac{-1}{2}, \frac{-1}{2}\sqrt{2k-1}\right)$

Since triangle is equilateral

$\therefore \quad \frac{1}{4}(2k-1)+\frac{1}{4}=(2k-1) \Rightarrow k=2/3$

45. (A) → (q), (B) → (r), (C) → (s), (D) → (q)

(A) $|z|=\frac{|z_1+i\bar{z}_2|}{|z_2+i\bar{z}_1|}=\frac{|i||-iz_1+\bar{z}_2|}{|z_2+iz_1|}=\frac{1\times|\bar{z}_2-iz_1|}{|\bar{z}_2-iz_1|}=1$

(B) Here, $\arg\left(\frac{z-2i}{z+2i}\right)=\frac{\pi}{2}$ or $-\frac{\pi}{2}$, So it represents a semicircle with diametric ends at points (0, 2) and (0, –2)

So, $|z|$ = radius = 2

(C) $(x + 6)^2 + y^2 = (2x + 3)^2 + (2y)^2$

$\Rightarrow x^2 + y^2 + 12x + 36 = 4x^2 + 4y^2 + 12x + 9$

$\Rightarrow 3x^2 + 3y^2 = 27 \Rightarrow x^2 + y^2 = 9, |z| = 3$

(D) $z=\frac{1}{w}\left(\frac{aw+bw^2+cw^3}{c+aw+bw^2}\right)+\frac{1}{w^2}\left(\frac{aw^2+bw^3+cw^4}{b+cw+aw^2}\right)$

$=\frac{1}{w}\times 1+\frac{1}{w^2}\times 1=w^2+w=-1 \Rightarrow |z|=1$

46. (5) : $|z_1 - z_0| = |z_2 - z_0| = |z_3 - z_0| = |\lambda|$

Now, $\frac{z_3-z_0}{z_2-z_0}=\frac{e^{i7\pi/11}}{e^{i\pi/4}}=e^{i17\pi/44}$

$\Rightarrow \angle BSC=17\frac{\pi}{44}$ (where S represents z_0)

$\Rightarrow \angle BAC=17\frac{\pi}{88}$

Similarly $\frac{z_2-z_0}{z_1-z_0}=e^{i\pi/4} \Rightarrow \angle ACB=\frac{\pi}{8}$

$\therefore \angle ABC = \pi-\frac{\pi}{8}-\frac{17\pi}{88}=\frac{15\pi}{22}$

47. (3) : $\tan^{-1}\frac{(a-b)y}{x^2+y^2-(a+b)x+ab}=\frac{\pi}{4}$

$\Rightarrow x^2 + y^2 - (a + b)x - (a - b)y + ab = 0$

Centre $=\frac{3+i}{2} \Rightarrow a+b=3$

48. (1) : Clearly, $\alpha_1 = 1$; $\alpha_2 = \alpha$; $\alpha_3 = \alpha^2$; $\alpha_4 = \alpha^3$; $\alpha_5 = \alpha^4$; where $\alpha = e^{i2\pi/5}$

$\therefore \lambda = \alpha_3^{1001} = (\alpha^2)^{1001} = \alpha^{2002} = \alpha^{5\times400+2} = \alpha^2$

$\mu = (\alpha_4)^{669+1/3} = (\alpha^3)^{(669+1/3)} = \alpha^{2008} = \alpha^3$

$\nu = (\alpha_5)^{503+1/2} = (\alpha^4)^{503+1/2} = \alpha^{2014} = \alpha^{5\cdot402+4} = \alpha^4$

Also sum of 2011$^{\text{th}}$ power of roots of unity is 0

So, $1 + \alpha^{2011} + \lambda^{2011} + \mu^{2011} + \nu^{2011} = 0$

$\lambda^{2011} + \mu^{2011} + \nu^{2011} = -(1 + \alpha^{2011})$

$\lambda^{2011} + \mu^{2011} + \nu^{2011} = -(1 + \alpha)$

$|\lambda^{2011} + \mu^{2011} + \nu^{2011}| = |-(1 + e^{i2\pi/5})|$

$= |1 + \cos 2\pi/5 + i \sin 2\pi/5| = |2 \cos \pi/5 (\cos \pi/5 + i \sin \pi/5)|$

$= 2\frac{\sqrt{5}+1}{4}=\frac{\sqrt{5}+1}{2}$ $\quad\therefore [|\lambda^{2011} + \mu^{2011} + \nu^{2011}|] = 1$

49. (1) : Solving the equation of the lines we get $z=-\bar{z} \Rightarrow z=i$

$|\alpha - i| = 2$; $\alpha = i \pm 2e^{i\theta}$, put it in the equation of the second line, we get $\cos\theta - \sin\theta = 0$

$\alpha=i\pm 2e^{\frac{i\pi}{4}}$

$\therefore \quad x=\pm\sqrt{2} \Rightarrow [|x|] = 1$

50. (2) : Let x be the (2009)$^{\text{th}}$ root of unity $\neq 1$, then

$x^{2009} - 1 = (x - 1)(x - w)....(x - w^{2008})$

Taking log on both sides, we get

$\ln (x^{2009} - 1) = \ln (x - 1) + \ln (x - w) + \ln (x - w^2) +....+ \ln (x - w^{2008})$

$\therefore$ Differentiating both the sides w.r.t. x, we get

$$\frac{(2009)x^{2008}}{x^{2009}-1}=\frac{1}{x-1}+\sum_{r=1}^{2008}\frac{1}{x-w^r} \quad ...(i)$$

Putting $x = 2$ in equation (i), we get

$$1+\sum_{r=1}^{2008}\frac{1}{2-w^r}=\frac{2009\left(2^{2008}\right)}{2^{2009}-1}$$

Multiplying both sides of above equation by $(2^{2009} - 1)$, we get

$\therefore \quad \left(2^{2009}-1\right)\sum_{r=1}^{2008}\frac{1}{2-w^r}=2009\cdot 2^{2008}-2^{2009}+1$

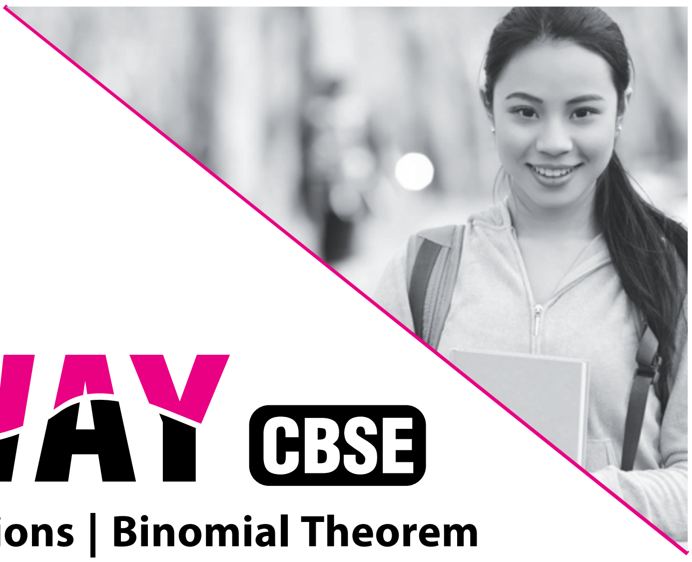
$= 2^{2008}(2009 - 2) + 1 = 2^{2008}\cdot 2007 + 1 = [(a)(2^b) + c]$

$\therefore \quad a = 2007, b = 2008, c = 1$

Hence $a + b + c = 4016$

CLASS XI | Series 5

ACE YOUR WAY CBSE



Permutations and Combinations | Binomial Theorem

IMPORTANT FORMULAE

PERMUTATIONS AND COMBINATIONS

- n factorial $= \lfloor n$ or $n! = n(n-1)(n-2) \ldots 2\cdot 1$
- $\lfloor 0 = 0! = 1$
- $^nP_r = P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$
  $= \dfrac{n!}{(n-r)!} (r < n)$
- $^nP_n = n(n-1) \ldots 2 \cdot 1 = n!$
- Number of permutations of n different things taken r at a time when
  - (i) repetition is allowed $= n^r$
  - (ii) a particular thing is included $= r \cdot {}^{n-1}P_{r-1}$
  - (iii) a particular thing is not included $= {}^{n-1}P_r$
- (i) No. of permutations when 'r' particular things are together $= (n-r+1)!\, r!$
  (ii) If 'r' particular things are identical then no. of permutations $= (n-r+1)!$
- No. of ways of arranging 'n' things when 'p' things are of one kind, 'q' things are of second kind and rest are all different $= \dfrac{n!}{p!q!}$
- $^nC_r = \dbinom{n}{r} = C(n,r) = \dfrac{n!}{r!(n-r)!} (r < n) = \dfrac{^nP_r}{r!}$
- $^nC_r = {}^nC_{n-r}$
- $^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$
- $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- $^nC_r = \dfrac{n}{r} {}^{n-1}C_{r-1}$
- No. of combinations of n different things taken 'r' at a time when 'p' particular things are always included $= {}^{n-p}C_{r-p}$
- No. of combinations when 'p' particular things are always excluded $= {}^{n-p}C_r$
- No. of ways of selecting zero or more things out of 'n' identical things $= 1 + 1 + 1 + \ldots$ upto $(n+1)$ terms $= n + 1$
- No. of ways of selecting zero or more things out of n different things $= {}^nC_0 + {}^nC_1 + \ldots + {}^nC_n = 2^n$

BINOMIAL THEOREM

- $(a + x)^n = {}^nC_0\, a^n x^0 + {}^nC_1 a^{n-1}x^1 + {}^nC_2\, a^{n-2}x^2 + \ldots\ldots + {}^nC_n\, a^0 x^n \;\forall\; n \in I^+$
- *General term in the expansion of* $(a + b)^n$ *is* T_{r+1} *i.e.,* $(r + 1)^{\text{th}}$ *term* $= {}^nC_r\, a^{n-r} \cdot b^r$
- *Middle term : In the expansion of* $(a + b)^n$*, the middle term is given by*
  - $\left(\frac{n}{2}+1\right)^{\text{th}}$ *term, if n is even*
  - $\left(\frac{n+1}{2}\right)^{\text{th}}$ *term and* $\left(\frac{n+1}{2}+1\right)^{\text{th}}$ *term, if n is odd*
- *Number of terms in the binomial expansion of* $(a + x)^n = n + 1$
- *In the expansion of* $(x + y)^n$
  - *(i)* r^{th} *term from end* $= (n - r + 2)^{th}$ *term from beginning*
  - *(ii)* r^{th} *term from end* $= r^{th}$ *term from beginning in the expansion of* $(y + x)^n$
- *Some Important Expansions*
  - $(1 + x)^n = 1 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots\ldots + {}^nC_n x^n = \sum_{r=0}^{n} {}^nC_r x^r$
  - $(1 - x)^n = 1 - {}^nC_1\, x + {}^nC_2\, x^2 - \ldots\ldots + (-1)^n\, {}^nC_n x^n = \sum_{r=0}^{n} (-1)^r . {}^nC_r x^r$
  - $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots + (-1)^r x^r + \ldots\ldots \infty$
  - $(1 + x)^{-2} = 1 - 2x + 3x^2 - \ldots\ldots + (-1)^r (r + 1)\, x^r + \ldots\ldots \infty$
  - $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots + x^r + \ldots\ldots \infty$
  - $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots + (r + 1)\, x^r + \ldots\ldots \infty$

WORK IT OUT

VERY SHORT ANSWER TYPE

1. If $(n + 2)! = 2550\ (n!)$, find n.
2. Prove that : $(2n)! = 2^n\ (n!)\ [1{\cdot}3{\cdot}5{\cdot}\ \ldots\ {\cdot}(2n-1)]$
3. Find the $(n+1)^{\text{th}}$ term from the end in the expansion of $\left(x-\frac{1}{x}\right)^{3n}$.
4. How many 4-letter words, with or without meaning, can be formed out of the letters of the word 'LOGARITHMS' if repetition of letters is not allowed?
5. Expand $\left(x-\frac{1}{y}\right)^{11}$, $y \neq 0$

SHORT ANSWER TYPE

6. Find the middle terms in the expansion of $\left(3x-\frac{x^3}{6}\right)^{7}$.
7. Find the greatest term in $(5 + 2y)^{17}$ if $y=\frac{1}{2}$.
8. Find r, if $5P(4, r) = 6P(5, r - 1)$.
9. For a set of five true or false questions, no student has written all correct answer and no two students have given the same sequence of answers. What is the maximum number of students in the class for this to be possible?
10. Find
    (i) Coefficient of x^2 in $(1 - 2x + 3x^2)\ (1 - x)^{14}$
    (ii) Coefficient of x^{11} in the expansion of $(1 - 2x + 3x^2)\ (1 + x)^{11}$.

LONG ANSWER TYPE - I

11. If $C(n, r) : C(n, r + 1) = 1 : 2$ and $C(n, r + 1) : C(n, r + 2) = 2 : 3$, determine the values of n and r.
12. Assuming that x is so small that its second and higher powers may be neglected, simplify the following: $\dfrac{(1-2x)^{2/3}(4+5x)^{3/2}}{\sqrt{1-x}}$
13. If x^p occurs in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$, prove that its coefficient is $\dfrac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}$
14. The letters of the word 'OUGHT' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'TOUGH' in this dictionary.
15. In how many ways can 9 examination papers be arranged so that the best and the worst papers are never together?

LONG ANSWER TYPE - II

16. In how many ways can 3 prizes be distributed among 4 boys, when
(i) no boy gets more than 1 prize
(ii) a boys may get any number of prizes
(iii) no boy gets all the prizes?

17. Find a, b and n in the expansion of $(a - b)^n$ if the first three terms of the expansion are 729, 7290 and 30375 respectively.

18. How many numbers are there between 100 and 1000 such that at least one of their digits is 7?

19. If a_1, a_2, a_3 and a_4 be any four consecutive coefficients in the expansion of $(1 + x)^n$, prove that

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}$$

20. If p is nearly equal to q and $n > 1$, show that

$$\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}=\left(\frac{p}{q}\right)^{1/n}.$$

Hence, find the approximate value of $\left(\frac{99}{101}\right)^{1/6}$.

SOLUTIONS

1. $(n + 2)! = 2550\ (n!)$
$\Rightarrow$ $(n + 2)\ (n + 1)\ (n!) = 2550\ (n!)$
$\Rightarrow$ $(n + 2)\ (n + 1) = 2550 \Rightarrow n^2 + 3n - 2548 = 0$
$\Rightarrow$ $(n - 49)\ (n + 52) = 0$
$\Rightarrow$ $n = 49$ as $n = -52$ is rejected being $n \in N$.
$\therefore$ $n = 49$

2. $(2n)! = (2n)(2n-1)(2n-2)(2n-3) \ldots 4\cdot3\cdot2\cdot1$
$= [(2n)(2n-2)(2n-4) \ldots 4\cdot2][(2n-1)(2n-3) \ldots 3\cdot1]$
$= 2^n \times [n(n-1)(n-2) \ldots 2\cdot1] \times [(2n-1)(2n-3) \ldots 3\cdot1]$
$= 2^n \cdot (n!) \cdot [1\cdot3\cdot5 \cdot \ldots (2n-1)]$.

3. Given expansion is $\left(x-\frac{1}{x}\right)^{3n}$

$(n + 1)^{\text{th}}$ term from the end in the given expansion $= (3n - n + 1)^{\text{th}}$ *i.e.*, $(2n + 1)^{\text{th}}$ term from the beginning in the same expansion .

$\therefore$ Required term $= T_{2n+1} = {}^{3n}C_{2n}x^{3n-2n}\left(-\frac{1}{x}\right)^{2n}$

$$=\frac{3n!}{(2n)!n!}x^n\frac{(-1)^{2n}}{x^{2n}}=\frac{3n!}{(2n)!n!}\frac{1}{x^n}.$$

4. The word 'LOGARITHMS' contains 10 different letters.
$\therefore$ Number of required words = number of arrangements of 10 letters, taken 4 at a time
$= {}^{10}P_4 = 10 \times 9 \times 8 \times 7 = 5040$

5. $$\left(x-\frac{1}{y}\right)^{11}={}^{11}C_0x^{11}\left(\frac{1}{y}\right)^0-{}^{11}C_1x^{10}\left(\frac{1}{y}\right)+{}^{11}C_2x^9\left(\frac{1}{y}\right)^2-{}^{11}C_3x^8\left(\frac{1}{y}\right)^3+{}^{11}C_4x^7\left(\frac{1}{y}\right)^4$$
$$-{}^{11}C_5x^6\left(\frac{1}{y}\right)^5+{}^{11}C_6x^5\left(\frac{1}{y}\right)^6-{}^{11}C_7x^4\left(\frac{1}{y}\right)^7+{}^{11}C_8x^3\left(\frac{1}{y}\right)^8-{}^{11}C_9x^2\left(\frac{1}{y}\right)^9+{}^{11}C_{10}x\left(\frac{1}{y}\right)^{10}-{}^{11}C_{11}\left(\frac{1}{y}\right)^{11}$$
$$=x^{11}-11\frac{x^{10}}{y}+55\frac{x^9}{y^2}-165\frac{x^8}{y^3}+330\frac{x^7}{y^4}-462\frac{x^6}{y^5}+462\frac{x^5}{y^6}-330\frac{x^4}{y^7}+165\frac{x^3}{y^8}-55\frac{x^2}{y^9}+\frac{11x}{y^{10}}-\frac{1}{y^{11}}$$

6. Clearly, the given expansion contains 8 terms

$\therefore$ Middle terms are $\left(\frac{8}{2}\right)^{\text{th}}$ and $\left(\frac{8}{2}+1\right)^{\text{th}}$ terms, *i.e*,, 4^{th} and 5^{th} terms

Now, $t_4 = t_{3+1} = (-1)^3\ {}^7C_3\cdot(3x)^{(7-3)}\cdot\left(\frac{x^3}{6}\right)^3$

$$=\left(-35\times81\times x^4\times\frac{x^9}{216}\right)=\frac{-105x^{13}}{8}.$$

And, $t_5 = t_{4+1} = (-1)^4\cdot{}^7C_4\cdot(3x)^{(7-4)}\cdot\left(\frac{x^3}{6}\right)^4$

$$=\left(35\times27\times x^3\times\frac{x^{12}}{1296}\right)=\frac{35x^{15}}{48}.$$

7. Here, $\frac{T_{r+1}}{T_r}=\frac{17-r+1}{r}\cdot\frac{2y}{5}=\frac{18-r}{5r}\quad\left(\because\ y=\frac{1}{2}\right)$

$\therefore$ $T_{r+1} >, =$ or $< T_r$ according as $\frac{18-r}{5r} >, =$ or < 1,

Now, $18 - r = 5r \Rightarrow 18 = 6r \Rightarrow r = 3$

Hence the 3^{rd} and the 4^{th} terms are equal when $y=\frac{1}{2}$ $\therefore$ $T_3 = T_4 = {}^{17}C_2(5)^{15} = 17 \times 8 \times (5)^{15}$.

8. $5P(4, r) = 6P(5, r-1)$

$$\Rightarrow\ 5\cdot\frac{4!}{(4-r)!} = 6\cdot\frac{5!}{[5-(r-1)]!}$$

$$\Rightarrow\ \frac{5!}{(4-r)!} = \frac{6\times 5!}{(6-r)!}$$

$$\Rightarrow\ \frac{1}{(4-r)!} = \frac{6}{(6-r)(5-r)(4-r)!}$$

$\Rightarrow\ (6-r)(5-r) = 6 \Rightarrow r^2 - 11r + 30 = 6$

$\Rightarrow (r-3)(r-8) = 0$

$\Rightarrow\ r = 3$ or 8

Since $P(n, r)$ is defined only when $r \le n$, we reject $r = 8$. Hence $r = 3$.

9. Clearly, there are 2 ways of answering each of the 5 questions, *i.e.*, true for false.

$\therefore$ Total number of different sequences of answers $= 2\times 2\times 2\times 2\times 2 = 32$.

There is only one all-correct answer.

So, the maximum number of sequences leaving all-correct answer is $(32-1) = 31$.

Since different students have given different sequences of answers, so the maximum possible number of students = 31.

10. **(i)** We have, $(1-2x+3x^2)(1-x)^{14} = (1-2x+3x^2)\times(1 - {}^{14}C_1x + {}^{14}C_2x^2 - \ldots$ to 15 terms)

$\therefore$ Co-efficient of $x^2 = {}^{14}C_2 - 2\times{}^{14}C_1 + 3 = 66$

(ii) we have $(1-2x+3x^2)(1+x)^{11} = (1-2x+3x^2)\times(1 + {}^{11}C_1x + {}^{11}C_2x^2 + \ldots + {}^{11}C_9x^9 + {}^{11}C_{10}x^{10} + x^{11})$

$\therefore$ Coefficient of $x^{11} = 1\times 1 - 2\times{}^{11}C_{10} + 3\times{}^{11}C_9$

$$1 - 2\times 11 + 3\times\frac{11\times 10}{2!} = 144$$

11. $C(n, r) : C(n, r+1) = 1 : 2$

$$\Rightarrow\ \frac{n!}{r!(n-r)!} : \frac{n!}{(r+1)!(n-r-1)!} = 1:2$$

$$\Rightarrow\ \frac{n!}{r!(n-r)(n-r-1)!}\times\frac{(r+1)r!(n-r-1)!}{n!} = \frac{1}{2}$$

$$\Rightarrow\ \frac{r+1}{n-r} = \frac{1}{2}\ \Rightarrow\ n - 3r - 2 = 0 \qquad \ldots\text{(i)}$$

Also $C(n, r+1) : C(n, r+2) = 2:3$

$$\Rightarrow\ \frac{n!}{(r+1)!(n-r-1)!} : \frac{n!}{(r+2)!(n-r-2)!} = 2:3$$

$$\Rightarrow\ \frac{n!}{(r+1)!(n-r-1)(n-r-2)!}\times\frac{(r+2)(r+1)!(n-r-2)!}{n!} = \frac{2}{3}$$

$$\Rightarrow\ \frac{r+2}{n-r-1} = \frac{2}{3}\ \Rightarrow\ 2n - 5r - 8 = 0 \qquad \ldots\text{(ii)}$$

Solving (i) and (ii) we get, $n = 14$, $r = 4$

12. Using binomial expansion, we have

$$\frac{(1-2x)^{2/3}(4+5x)^{3/2}}{\sqrt{1-x}}$$

$$= (1-2x)^{2/3}\cdot 4^{3/2}\left(1+\frac{5}{4}x\right)^{3/2}\cdot(1-x)^{-1/2}$$

$$= 8(1-2x)^{2/3}\cdot\left(1+\frac{5}{4}x\right)^{3/2}\cdot(1-x)^{-1/2}$$

$$= 8\times\left[\left\{1+\frac{2}{3}\times(-2x)+\ldots\right\}\times\left\{1+\frac{3}{2}\times\frac{5}{4}x+\ldots\right\}\times\left\{1+\left(-\frac{1}{2}\right)(-x)+\ldots\right\}\right]$$

$$= 8\times\left[\left(1-\frac{4}{3}x\right)\left(1+\frac{15}{8}x\right)\left(1+\frac{1}{2}x\right)\right]$$

[neglecting x^2 and higher powers of x]

$$= 8\times\left[\left(1-\frac{4}{3}x+\frac{15}{8}x\right)\left(1+\frac{1}{2}x\right)\right] \quad \text{[neglecting } x^2\text{]}$$

$$= 8\times\left[\left(1+\frac{13}{24}x\right)\left(1+\frac{1}{2}x\right)\right] = 8\times\left[1+\frac{1}{2}x+\frac{13}{24}x\right]$$

[neglecting x^2]

$$= 8\times\left(1+\frac{25}{24}x\right) = \left(8+\frac{25}{3}x\right).$$

13. The general term in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$ is :

$$T_{r+1} = {}^{2n}C_r(x^2)^{2n-r}\cdot\left(\frac{1}{x}\right)^r$$

$$= {}^{2n}C_r\,x^{4n-2r}\cdot x^{-r} = {}^{2n}C_r\,x^{4n-3r}$$

This contains x^p if $4n - 3r = p\ \Rightarrow\ r = \frac{4n-p}{3}$

$\therefore$ Co-efficient of $x^p = {}^{2n}C_r$ where $r = \frac{4n-p}{3}$

$$= \frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(2n-\frac{4n-p}{3}\right)!} = \frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}$$

14. Total number of letters in the word OUGHT is 5 and all the five letters are different. Alphabetical order of letters is G, H, O, T, U.

Number of words beginning with G = 4! = 24
Number of words beginning with H = 4! = 24
Number of words beginning with O = 4! = 24
Number of words beginning with TG = 3! = 6
Number of words beginning with TH = 3! = 6
Number of words beginning with TOG = 2! = 2
Number of words beginning with TOH = 2! = 2
There will be two words beginning with TOU and TOUGH is first among them

$\therefore$ Rank of word 'TOUGH' in the dictionary
= 24 + 24 + 24 + 6 + 6 + 2 + 2 + 1 = 89th

15. The number of ways of arranging 9 papers = ${}^9P_9 = 9\,!$
Let us first consider the number of permutations, each containing the best and the worst papers together.
Let us tie these two papers together and consider them as one paper.
Then, these 8 papers can be arranged in ${}^8P_8 = 8\,!$ ways.
Also, these two papers may be arranged among themselves in 2! *i.e.*, 2 ways.
So, the number of permutations with the worst and the best papers together = (2 × 8!).
$\therefore$ The number of arrangements with best and worst papers never together = [(9!) –(2 × 8!)]
= (9 – 2) × 8! = 7 × 8 ! = 282240

16. (i) The first prize can be given away in 4 ways as it may be give to anyone of the 4 boys.
So, the second prize can be given away in 3 ways, and third prize can be given away to anyone of the remaining 2 boys in 2 ways.
$\therefore$ The of ways in which all the prizes can be given away = (4 × 3 × 2) = 24.

(ii) The first prize can be given away in 4 ways as it may be given to anyone of the 4 boys.
The second prize can be given away in 4 ways, because there is no restriction as to the number of prizes a student gets.
Similarly, the third prize can be given away in 4 ways. Hence, the number of ways in which all the prizes can be given away = (4 × 4 × 4) = 64.

(iii) The number of ways in which a boy gets all the 3 prizes is 4, as anyone of the 4 boys may get all the prizes.
$\therefore$ The number of ways in which a boy does not get all the prizes = 64 – 4 = 60

17. T_1 of $(a+b)^n = a^n = 729$ ...(i)
T_2 of $(a+b)^n = {}^nC_1 a^{n-1} b = 7290$ ...(ii)
T_3 of $(a+b)^n = {}^nC_2 a^{n-2} b^2 = 30375$ ...(iii)

Dividing (i) by (ii), we get

$$\frac{a^n}{{}^nC_1 a^{n-1} b} = \frac{729}{7290} = \frac{1}{10} \Rightarrow \frac{a}{nb} = \frac{1}{10} \Rightarrow \frac{a}{b} = \frac{n}{10} \quad \text{...(iv)}$$

Dividing (ii) by (iii), we get

$$\frac{{}^nC_1 a^{n-1} b}{{}^nC_2 a^{n-2} b^2} = \frac{7290}{30375}$$

$$\Rightarrow \frac{n a^{n-1} b}{\frac{n(n-1)}{2} a^{n-2} b^2} = \frac{7290}{30375} = \frac{6}{25}$$

$$\Rightarrow \frac{2}{n-1} \times \frac{a}{b} = \frac{6}{25} \quad \text{...(v)}$$

$$\frac{2}{n-1} \times \frac{n}{10} = \frac{6}{25} \text{ (using (iv))} \Rightarrow n = 6$$

Putting $n = 6$ in (i) we get, $a^6 = 729 \Rightarrow a = 3$
Putting $n = 6$, $a = 3$ in (iv), we get $b = 5$

18. We have to form 3-digit numbers such that at least one of their digits is 7. Now,

(i) 3-digit numbers with 7 at the unit's place:
The number of ways to fill the hundred's place = 9 [by any digit from 1 to 9].
The number of ways to fill the ten's place = 10 [by any digit from 0 to 9].
The number of ways to fill the unit's place = 1 [by 7 only]
$\therefore$ Number of such numbers = 9 × 10 × 1 = 90.

(ii) 3-digit numbers with 7 at the ten's place but not at unit's place :
The number of ways to fill the hundred's place = 9 [by any digit from 1 to 9].
The number of ways to fill the ten's place = 1 [by 7 only]
The number of ways to fill the unit's place = 9 [by any digit from 0, 1, 2, 3, 4, 5, 6, 8, 9].
$\therefore$ Number of such numbers = 9 × 1 × 9 = 81.

MPP-5 CLASS XI ANSWER KEY

1.	(b)	**2.**	(c)	**3.**	(d)	**4.**	(c)	**5.**	(c)
6.	(d)	**7.**	(a,b,c)	**8.**	(b,c)	**9.**	(a,b,c)	**10.**	(d)
11.	(b, c)	**12.**	(a,b,c,d)			**13.**	(a,d)	**14.**	(c)
15.	(b)	**16.**	(a)	**17.**	(4)	**18.**	(5)	**19.**	(3)
20.	(8)								

(iii) 3-digit numbers with 7 at the hundred's place but neither at the unit's place nor at ten's place:

The number of ways to fill the hundred's place = 1. [by 7 only].

The number of ways to fill the ten's place = 9. [by any digit from 0, 1, 2, 3, 4, 5, 6, 8, 9].

The number of ways to fill the unit's place = 9 [by any digit from 0, 1, 2, 3, 4, 5, 6, 8, 9].

$\therefore$ Number of such numbers = 1 × 9 × 9 = 81.

Hence, the number of required type of numbers = (90 + 81 + 81) = 252.

19. Given, a_1, a_2, a_3 and a_4 are the coefficients of the r^{th}, $(r+1)^{th}$, $(r+2)^{th}$ and $(r+3)^{th}$ terms respectively in the expansion of $(1+x)^n$

Now, $a_1 = r^{th}$ coefficient $= {}^nC_{r-1}$ ...(i)

$a_2 = (r+1)^{th}$ coefficient $= {}^nC_r$ ...(ii)

$a_3 = (r+2)^{th}$ coefficient $= {}^nC_{r+1}$ ...(iii)

$a_4 = (r+3)^{th}$ coefficient $= {}^nC_{r+2}$ ...(iv)

Now, $$\frac{a_2}{a_1} = \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n!}{r!(n-r)!} \cdot \frac{(r-1)!(n-r+1)!}{n!} = \frac{n-r+1}{r}$$

$$\therefore \quad \frac{a_2+a_1}{a_1} = \frac{a_2}{a_1}+1 = \frac{n-r+1}{r}+1 = \frac{n+1}{r}$$

$$\therefore \quad \frac{a_1}{a_1+a_2} = \frac{r}{n+1} \quad \text{...(v)}$$

Putting $r+1$ in place of r in (v), we get

$$\frac{a_2}{a_2+a_3} = \frac{r+1}{n+1} \quad \text{...(vi)}$$

Similarly, $$\frac{a_3}{a_3+a_4} = \frac{r+2}{n+1} \quad \text{...(vii)}$$

$$\therefore \quad \frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{r}{n+1} + \frac{r+2}{n+1} = \frac{2(r+1)}{n+1} = \frac{2a_2}{a_2+a_3}$$

20. Let $p = q + h$, where h is so small that its square and higher powers may be neglected. Then,

$$\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} = \frac{(n+1)(q+h)+(n-1)q}{(n-1)(q+h)+(n+1)q}$$

$$= \frac{2nq+(n+1)h}{2nq+(n-1)h} = \frac{\left[1+\left(\frac{n+1}{2nq}\right)h\right]}{\left[1+\left(\frac{n-1}{2nq}\right)h\right]}$$

[on dividing num. and denom. by $2nq$]

$$= \left[1+\left(\frac{n+1}{2nq}\right)h\right]\left[1+\left(\frac{n-1}{2nq}\right)h\right]^{-1}$$

$$= \left[1+\left(\frac{n+1}{2nq}\right)h\right]\left[1-\left(\frac{n-1}{2nq}\right)h\right]$$

[expanding the 2nd expression and neglecting h^2, h^3, etc]

$$= \left[1+\left(\frac{n+1}{2nq}\right)h - \left(\frac{n-1}{2nq}\right)h\right] \quad \text{[neglecting } h^2\text{]}$$

$$= \left(1+\frac{h}{nq}\right).$$

Also, $$\left(\frac{p}{q}\right)^{1/n} = \left(\frac{q+h}{q}\right)^{1/n} = \left(1+\frac{h}{q}\right)^{1/n} = \left(1+\frac{h}{nq}\right).$$

[Expanding and neglecting h^2 and higher powers of h]

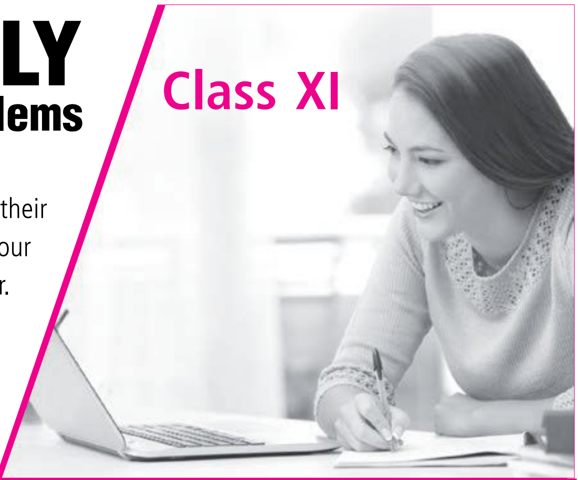
$$\therefore \quad \frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} = \left(\frac{p}{q}\right)^{1/n}$$

Deduction : Taking $p = 99$, $q = 101$ and $n = 6$ in the above result, we get

$$\left(\frac{99}{101}\right)^{1/6} = \frac{(6+1)\times 99+(6-1)\times 101}{(6-1)\times 99+(6+1)\times 101} = \frac{1198}{1202} = \frac{599}{601}.$$

MPP-5 MONTHLY Practice Problems

Class XI



This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Binomial Theorem

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. If O be the sum of odd terms and E that of even terms in the expansion of $(x + a)^n$, then $O^2 - E^2 =$
   (a) $(x^2 + a^2)^n$ (b) $(x^2 - a^2)^n$
   (c) $(x - a)^{2n}$ (d) none of these

2. $(115)^{96} - (96)^{115}$ is divisible by
   (a) 15 (b) 17
   (c) 19 (d) 21

3. The number of terms in the expansion of $(a + b + c)^{10}$ is
   (a) 11 (b) 21
   (c) 55 (d) 66

4. The coefficient of the term independent of x in the expansion of $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$, is
   (a) $\frac{1}{3}$ (b) $\frac{19}{54}$ (c) $\frac{17}{54}$ (d) $\frac{1}{4}$

5. The sum $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$, where $\binom{p}{q} = 0$ if $p < q$, is maximum when m is
   (a) 5 (b) 10
   (c) 15 (d) 20

6. If $C_0, C_1, C_2, \ldots\ldots, C_n$ denote the coefficients in the expansion of $(1 + x)^n$, then
   $C_0 + 3 \cdot C_1 + 5 \cdot C_2 + \ldots\ldots + (2n + 1)C_n =$
   (a) $n \cdot 2^n$ (b) $(n - 1)2^n$
   (c) $(n + 1)2^{n-1}$ (d) $(n + 1)2^n$

One or More Than One Option(s) Correct Type

7. If $(1 + x - 2x^2)^6 = 1 + a_1x + a_2x^2 + a_{12}x^{12}$, then
   (a) $a_2 + a_4 + a_6 + \ldots\ldots + a_{12} = 31$
   (b) $a_1 + a_3 + a_5 + \ldots\ldots + a_{11} = -32$
   (c) $a_1 + a_2 + a_3 + \ldots\ldots + a_{12} = -1$
   (d) none of these

8. Which of the following is an even positive integer?
   (a) $(\sqrt{3}+1)^{2n+1}+(\sqrt{3}-1)^{2n+1}$
   (b) $(\sqrt{3}+1)^{2n}+(\sqrt{3}-1)^{2n}$
   (c) $(\sqrt{3}+1)^{2n+1}-(\sqrt{3}-1)^{2n+1}$
   (d) none of these

9. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, then
   (a) $C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n2^{n-1}$
   (b) $C_0 + 2C_1 + 3C_2 + \ldots + (n + 1)\, C_n = 2^{n-1}\,(n + 2)$
   (c) $C_0+\frac{C_1}{2}+\frac{C_2}{3}+\ldots+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$
   (d) none of these

10. $\sum_{r=1}^{n}\left(\sum_{p=0}^{r-1}{}^nC_r\,{}^rC_p\,2^p\right)$ is equal to
    (a) $4^n - 3^n + 1$ (b) $4^n - 3^n - 1$
    (c) $4^n - 3^n + 2$ (d) $4^n - 3^n$

11. In the expansion of $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$
    (a) the number of rational terms = 4
    (b) the number of irrational terms = 19
    (c) the middle term is irrational
    (d) the number of irrational terms = 17

12. Coefficient of x^n in the expansion of $(1 + x)^{2n}$ is equal to

(a) $\dfrac{P(2n,n)}{\lfloor n}$

(b) $\dfrac{(n+1)(n+2)(n+3).....(2n)}{\lfloor n}$

(c) $C(2n, n)$

(d) $\dfrac{1\cdot3\cdot5.....\cdot(2n-1)}{\lfloor n}2^n$

13. If n is a positive integer and $(3\sqrt{3}+5)^{2n+1}=\alpha+\beta$, where α is an integer and $0 < \beta < 1$, then

(a) α is an even integer

(b) $(\alpha + \beta)^2$ is divisible by 2^{2n+1}

(c) the integer just below $(3\sqrt{3}+5)^{2n+1}$ divisible by 3

(d) α is divisible by 10

Comprehension Type

Consider $(1 + x + x^2)^{2n} = \sum_{r=0}^{4n} a_r x^r$, where $a_0, a_1, a_2, ..., a_{4n}$ are real numbers and n is a positive integer.

14. The value of $\sum_{r=0}^{n-1} a_{2r}$ is

(a) $\dfrac{9^n - 2a_{2n} - 1}{4}$ (b) $\dfrac{9^n + 2a_{2n} + 1}{4}$

(c) $\dfrac{9^n - 2a_{2n} + 1}{4}$ (d) $\dfrac{9^n + 2a_{2n} - 1}{4}$

15. The value of $\sum_{r=1}^{n} a_{2r-1}$ is

(a) $\left(\dfrac{9^n-1}{2}\right)$ (b) $\left(\dfrac{3^{2n}-1}{4}\right)$

(c) $\left(\dfrac{3^{2n}+1}{4}\right)$ (d) $\left(\dfrac{9^n+1}{2}\right)$

Matrix Match Type

16. Match the following :

	Column I		Column II
P.	If last digit of the number 9^{9^9} is λ and last digit of $2^{\lambda^{100}}$ is μ, then	1.	$\lambda + \mu = 6$
Q.	If last digit of the number 2^{999} is λ and last digit of $3^{\lambda\lambda\lambda}$ is μ, then	2.	$\lambda + \mu = 11$
R.	Let $a = \dfrac{72!}{(36!)^2} - 1$ is divisible by $10\lambda + \mu$, then	3.	$\lambda - \mu = 4$
		4.	$\lambda^{\mu} + \mu^{\lambda} = 9$

	P	Q	R
(a)	2	4	3
(b)	3	4	1
(c)	1	4	3
(d)	2	1	3

Integer Answer Type

17. The remainder when 9^{103} is divided by 25 is equal to

18. If $(2r + 3)^{th}$ and $(r - 1)^{th}$ terms in the expansion of $(1 + x)^{15}$ have equal coefficients then r is

19. If the coefficient of the middle term in the binomial expansions of $(1 + \alpha x)^4$ and $(1 - \alpha x)^6$ is same, then -10α equals

20. If ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$, then, the least value of n would be

◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ......
No. of questions correct ......
Marks scored in percentage ......

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CONCEPT BOOSTERS

Functions

Class XII

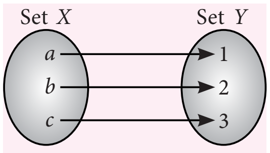
This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR, B.Tech, IIT Kanpur

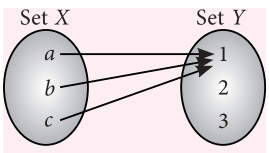
FUNCTIONS

Functions can be easily defined with the help of concept mapping. Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y". Let the correspondence be 'f' then mathematically we write $f: X \to Y$ where $y = f(x)$, $x \in X$ and $y \in Y$. We say that 'y' is the image of 'x' under f (or x is the pre image of y).

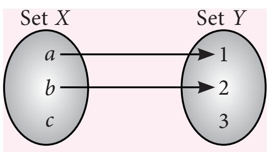
- A mapping $f: X \to Y$ is said to be a function if each element in the set X has its image in set Y. It is also possible that there are few elements in set Y which are not the images of any element in set X.
- Every element in set X should have one and only one image. That means it is impossible to have more than one image for a specific element in set X. Functions can not be multi-valued (A mapping that is multi-valued is called a relation from X and Y) *e.g.*



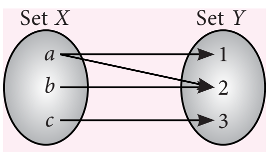
Function



Function

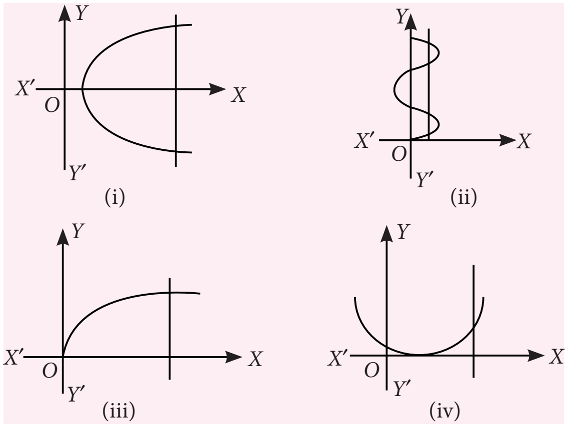


Not Function



Not Function

- **Testing for a function by vertical line test :** A relation $f: A \to B$ is a function or not, it can be checked by a graph of the relation. If it is possible to draw a vertical line which cuts the given curve at more than one point then the given relation is not a function and when this vertical line means line parallel to Y-axis cuts the curve at only one point then it is a function. Figure (iii) and (iv) represents a function.



- **Number of functions :** Let X and Y be two finite sets having m and n elements respectively. Then each element of set X can be associated to any one of n elements of set Y. So, total number of functions from set X to set Y is n^m.
- **Real valued function :** If R, be the set of real numbers and A, B are subsets of R, then the function $f: A \to B$ is called a real function or real valued function.

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

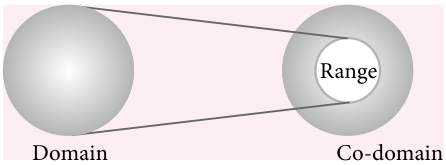
DOMAIN, CO-DOMAIN AND RANGE OF FUNCTION

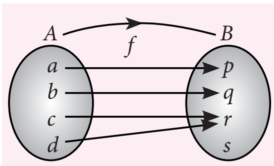
If a function f is defined from a set A to set B then ($f : A \to B$) set A is called the domain of f and set B is called the co-domain of f. The set of all f-images of the elements of A is called the range of f.

In other words, we can say

Domain = All possible values of x for which $f(x)$ exists.

Range = For all values of x, all possible values of $f(x)$.





Domain = $(a, b, c, d) = A$

Co-domain = $(p, q, r, s) = B$

Range = (p, q, r)

Methods for finding domain and range of function

- **Domain :** Expression under even root (*i.e.*, square root, fourth root etc.) ≥ 0. Denominator $\neq 0$.

  If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively then the domain of

  (i) $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ is $D_1 \cap D_2$.

  (ii) $\dfrac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x) = 0\}$

  (iii) $\left(\sqrt{f(x)}\right) = D_1 \cap \{x : f(x) \geq 0\}$

- **Range :** Range of $y = f(x)$ is collection of all outputs $f(x)$ corresponding to each real number in the domain.
  - If domain $\in$ finite number of points $\Rightarrow$ range $\in$ set of corresponding $f(x)$ values.
  - If domain $\in R$ or R – [some finite points]. Then express x in terms of y. From this find y for x to be defined (*i.e.*, find the values of y for which x exists).
  - If domain $\in$ a finite interval, find the least and greatest value for range using monotonicity.

- **Algebra of functions :** For functions $f : x \to R$; $g : x \to R$, we have $\forall\ x \in X$,
  - $(cf)(x) = cf(x)$, where c is a scalar.
  - $(f \pm g)(x) = f(x) \pm g(x)$.
  - $(fg)(x) = (gf)(x) = f(x)\, g(x)$.
  - $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, g(x) \neq 0$

- **Equal functions :** Two function f and g are said to be equal functions, if and only if

  (i) Domain of f = Domain of g

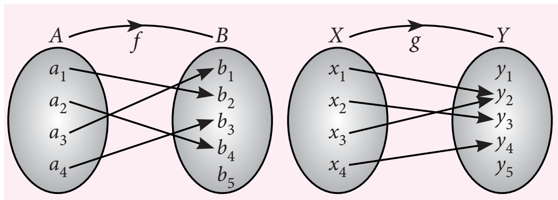
  (ii) Co-domain of f = Co-domain of g

  (iii) $f(x) = g(x)\ \forall\ x \in$ their common domain

KINDS OF FUNCTION

- **One-one function (injection) :** A function $f : A \to B$ is said to be a one-one function or an injection, if different elements of A have different images in B.

  e.g. Let $f : A \to B$ and $g : X \to Y$ be two functions represented by the following diagrams.



Clearly, $f : A \to B$ is a one-one function. But $g : X \to Y$ is not one-one function because two distinct elements x_1 and x_3 have the same image under function g.

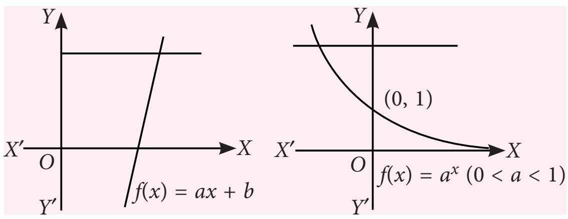
- **Method to check the injectivity of a function**

  (i) Take two arbitrary elements x, y (say) in the domain of f.

  (ii) Solve $f(x) = f(y)$. If $f(x) = f(y)$ gives $x = y$ only, then $f : A \to B$ is a one-one function (or an injection). Otherwise not.

  If function is given in the form of ordered pairs and if two ordered pairs do not have same second element then function is one-one.

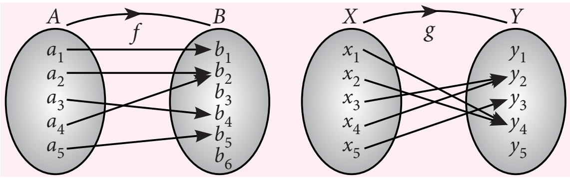
  If the graph of the function $y = f(x)$ is given and each line parallel to x-axis cuts the given curve at maximum one point then function is one-one. *e.g.*



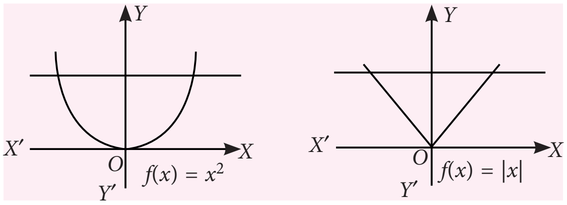
- **Number of one-one functions (injections) :** If A and B are finite sets having m and n elements respectively, then number of one-one functions from A to B = $\begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

- **Many-one function :** A function $f : A \to B$ is said to be a many-one function if two or more elements of set A have the same image in B.

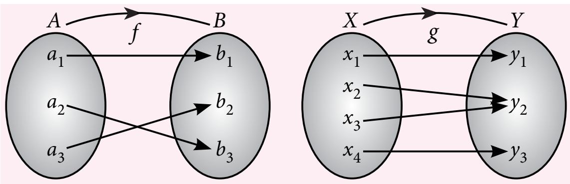
Thus, $f : A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.
In other words, $f : A \to B$ is a many-one function if it is not a one-one function.



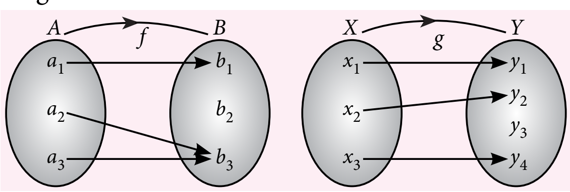
- If function is given in the form of set of ordered pairs and the second element of atleast two ordered pairs are same then function is many-one.
- If the graph of $y = f(x)$ is given and the line parallel to x-axis cuts the curve at more than one point then function is many-one.



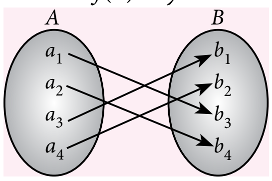
- **Onto function (surjection) :** A function $f : A \to B$ is onto if each element of B has its pre-image in A. In other words, Range of f = Co-domain of f. *e.g.* The following arrow-diagram shows onto function.



- **Number of onto function (surjection) :** If A and B are two sets having m and n elements respectively such that $1 \leq n \leq m$, then number of onto functions from A to B is $\sum_{r=1}^{n}(-1)^{n-r}\,{}^nC_r r^m$
- **Into function :** A function $f : A \to B$ is an into function if there exists an element in B having no pre-image in A.
In other words, $f : A \to B$ is an into function if it is not an onto function *e.g.,* The following arrow-diagram shows into function.



- **Method to find onto or into function :**
(i) Solve $f(x) = y$ by taking x as a function of y *i.e.,* $g(y)$(say).
(ii) Now if $g(y)$ is defined for each $y \in$ co-domain and $g(y) \in$ domain then $f(x)$ is onto and if any one of the above requirements is not fulfilled, then $f(x)$ is into.
- **One-one onto function (bijection) :** A function $f : A \to B$ is a bijection if it is one-one as well as onto.
In other words, a function $f : A \to B$ is a bijection if
(i) It is one-one *i.e.,* $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.
(ii) It is onto *i.e.,* for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.



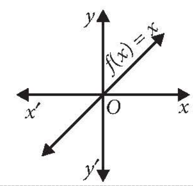
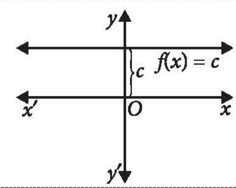
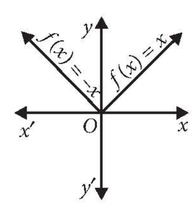
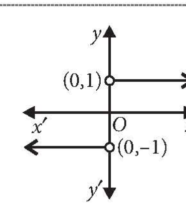
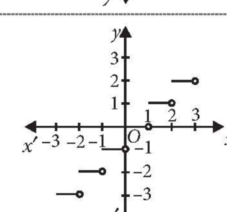
Clearly, f is a bijection since it is both injective as well as surjective.

- **Number of one-one onto function (bijection) :** If A and B are finite sets and $f : A \to B$ is a bijection, then A and B have the same number of elements. If A has n elements, then the number of bijection from A to B is the total number of arrangements of n items taken all at a time *i.e.* $n!$.
- **Algebraic functions :** Functions consisting of finite number of terms involving powers and roots of the independent variable and the four fundamental operations $+, -, \times$ and $\div$ are called algebraic functions.
e.g., (i) $x^{\frac{3}{2}} + 5x$ (ii) $\frac{\sqrt{x+1}}{x-1}, x \neq 1$
(iii) $3x^4 - 5x + 7$
- **Transcendental function :** A function which is not algebraic is called a transcendental function. *e.g.,* trigonometric; inverse trigonometric , exponential and logarithmic functions are all transcendental functions.
- **Trigonometric function :** A function is said to be a trigonometric function if it involves circular functions (sine, cosine, tangent, cotangent, secant, cosecant) of variable angles.

CONCEPT MAP

RELATIONS AND FUNCTIONS

Class XI / XII

Name of Function	Definition	Domain	Range	Graph
1. Identity Function	The function $f: R \to R$ defined by $f(x) = x \ \forall\ x \in R$	R	R	
2. Constant Function	The function $f: R \to R$ defined by $f(x) = c \ \forall\ x \in R$	R	$\{c\}$	
3. Polynomial Function	The function $f: R \to R$ defined by $f(x) = p_0 + p_1x + p_2x^2 + \ldots\ldots + p_nx^n$, where $n \in N$ and $p_0, p_1, p_2, \ldots, p_n \in R$ $\forall\ x \in R$			
4. Rational Function	The function f defined by $f(x)=\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions, $Q(x) \neq 0$			
5. Modulus Function	The function $f: R \to R$ defined by $f(x)=\lvert x\rvert=\begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \quad \forall\ x \in R$	R	$[0, \infty)$	
6. Signum Function	The function $f: R \to R$ defined by $f(x)=\begin{cases} \dfrac{\lvert x\rvert}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \\ 0, & x = 0 \end{cases}$	R	$\{-1, 0, 1\}$	
7. Greatest Integer Function	The function $f: R \to R$ defined by $f(x)=[x]=\begin{cases} x, \ x \in Z \\ \text{integer less than} \\ \text{equal to } x, \ x \notin Z \end{cases}$	R	Z	
8. Linear Function	The function $f: R \to R$ defined by $f(x) = mx + c$, $x \in R$ where m and c are constants	R	R	

RELATIONS

- R is a relation from A to B (where $A, B \neq \phi$) if $R \subseteq A \times B \Leftrightarrow R \subseteq \{(a, b) : a \in A, b \in B\}$
- Dom $(R) = \{a : (a, b) \in R\}$
- Range $(R) = \{b : (a, b) \in R \; \forall \; a \in A\}$

Cartesian Product of Sets

- Cartesian product of two sets A & B is denoted and defined as, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$
- Cartesian product of two sets is not commutative.
- **Note:**
  (i) If $n(A) = p$, $n(B) = q$, then $n(A \times B) = p \times q$.
  (ii) $(a, b) = (p, q) \Leftrightarrow a = p \;\&\; b = q$

Types of Relations

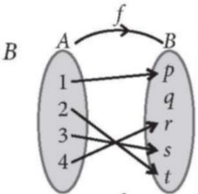
- **Empty (Void) Relation:** $R = \phi \Rightarrow R$ is void.
- **Universal Relation:** $R = A \times B \Rightarrow R$ is universal.
- **Reflexive Relation:** Every element is related to itself. *i.e.*, R is reflexive in $A \Leftrightarrow (a, a) \in R \; \forall \; a \in A$.
- **Symmetric Relation:** R is symmetric in A iff $(a, b) \in R \Rightarrow (b, a) \in R \; \forall \; a, b \in A$.
- **Transitive Relation:** R is transitive in A if $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R \; \forall \; a, b, c \in A$.
- **Equivalence Relation:** If R is reflexive, symmetric and transitive then R is equivalence.
- **Antisymmetric Relation:** R is antisymmetric if $(a, b) \in R, (b, a) \in R \Rightarrow a = b$.
- **Identity Relation:** $R = \{(a, a) \; \forall \; a \in A\}$ is an identity relation in A.
- **Inverse Relation:** R^{-1} is the inverse relation of R if $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$.
  Note: Dom (R) = Range (R^{-1})
  Range (R) = Dom (R^{-1})

FUNCTIONS

- A relation $(f : A \to B)$ where every element of set A has only one image in set B.
- Dom $(f) = \{a : (a, b) \in f\}$
- Range $(f) = \{b : (a, b) \in f \; \forall \; a \in A\}$

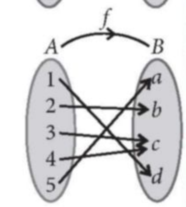
Types of Functions

One-one (Injective) Function



- No two elements of A have same image in B
  (i) $n(A) \leq n(B)$
  (ii) $f(x_1) = f(x_2)$
  $\Rightarrow x_1 = x_2$

Onto (Surjective) Function



- All the elements of B have atleast one pre-image in A.
  (i) $n(A) \geq n(B)$
  (ii) Range = Codomain

Bijective Function

- A function which is both one-one & onto.
  (i) $n(A) = n(B)$
  (ii) Range = Codomain
- If $n(A) = a = n(B)$, no. of bijections $= a!$s.

Composition of Functions

- A function gof, defined on the range of function f, is known as composition of functions.

$f : A \to B \quad g : B \to C$

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend right, "gof : A \to C"'] & B \arrow[r] & C \end{tikzcd}$$

Algebra of Functions

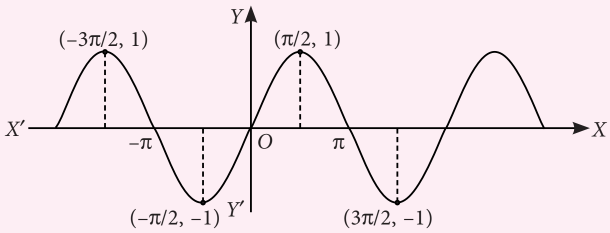
For $f : X \to R$ and $g : X \to R$

- $(f + g)(x) = f(x) + g(x), \; x \in X$
- $(kf)(x) = kf(x), \; x \in X$, k is constant
- $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, \; x \in X, g(x) \neq 0$
- $(f \cdot g)(x) = f(x) \cdot g(x), \; x \in X$
- $(f - g)(x) = f(x) - g(x), \; x \in X$

A binary operation $*$ on a set A is function $*$ from $A \times A \to A$

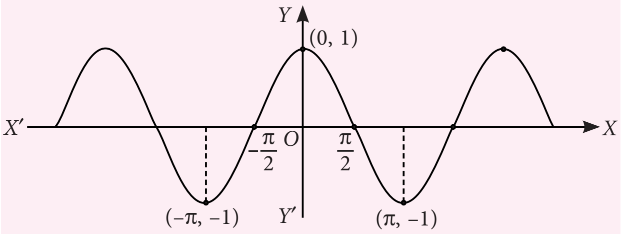
- $e \in X$ is an identity element if $a * e = a = e * a \; \forall \; a \in A$
- Commutative: $a * b = b * a \; \forall \; a, b \in A$
- $a \in A$ is invertible for $*$ if $\exists \; b \in A$ s.t. $a * b = e = b * a \; \forall \; a, b \in A$
- Associative: $(a * b) * c = a * (b * c) \; \forall \; a, b, c \in A$

- **Sine function**



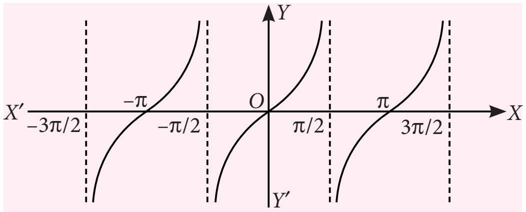
Domain = R ; Range = $[-1, 1]$

- **Cosine function**



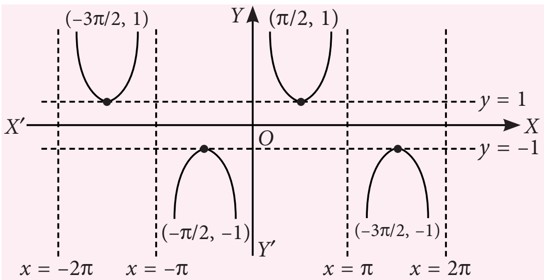
Domain = R; Range = $[-1, 1]$

- **Tangent function**



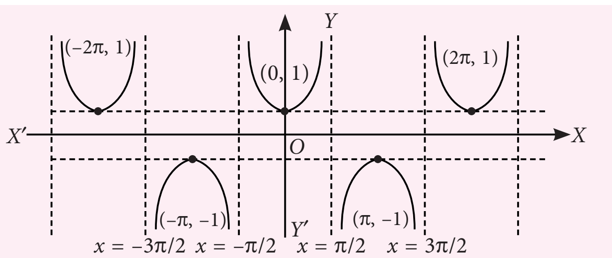
Domain = $R - \{(2n + 1)\ \pi/2\ |n \in I\}$; Range = R

- **Cosecant function**



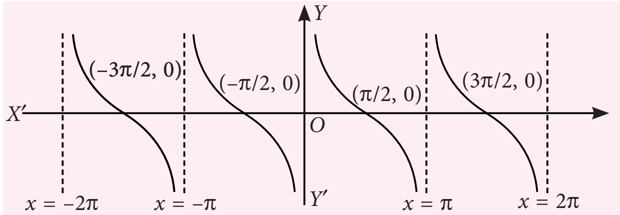
Domain = $R - \{(n\pi|n \in I\}$; Range = $(-\infty, -1] \cup [1, \infty)$

- **Secant function**



Domain = $R - \{(2n + 1)\ \pi/2\ |n \in I\}$; Range = $(-\infty, -1] \cup [1, \infty)$

- **Cotangent function**

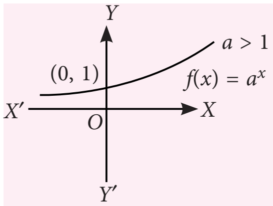


Domain = $R - \{n\pi\,|\,n \in I\}$; Range = R

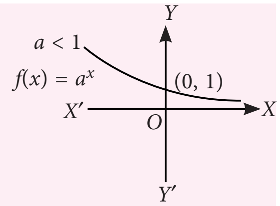
- **Inverse trigonometric functions**

Function	Domain	Range	Definition of the function
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$y = \sin^{-1} x \Leftrightarrow x = \sin y$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$y = \cos^{-1} x \Leftrightarrow x = \cos y$
$\tan^{-1} x$	$(-\infty, \infty)$ or R	$(-\pi/2, \pi/2)$	$y = \tan^{-1} x \Leftrightarrow x = \tan y$
$\cot^{-1} x$	$(-\infty, \infty)$ or R	$(0, \pi)$	$y = \cot^{-1} x \Leftrightarrow x = \cot y$
$\text{cosec}^{-1} x$	$R - (-1, 1)$	$[-\pi/2, \pi/2] - \{0\}$	$y = \text{cosec}^{-1} x \Leftrightarrow x = \text{cosec}\, y$
$\sec^{-1} x$	$R - (-1, 1)$	$[0, \pi] - \{\pi/2\}$	$y = \sec^{-1} x \Leftrightarrow x = \sec y$

- **Exponential function :** Let $a \neq 1$ be a positive real number. Then $f : R \to (0, \infty)$ defined by $f(x) = a^x$ called exponential function. Its domain is R and range is $(0, \infty)$.

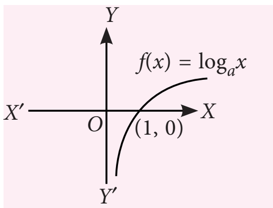


Graph of $f(x) = a^x$, when $a > 1$

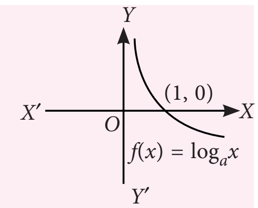


Graph of $f(x) = a^x$, when $a < 1$

- **Logarithmic function :** Let $a \neq 1$ be a positive real number. Then $f : (0, \infty) \to R$ defined by $f(x) = \log_a x$ is called logarithmic function. Its domain is $(0, \infty)$ and range is R.

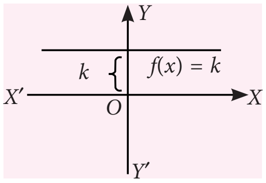


Graph of $f(x) = \log_a x$, when $a > 1$

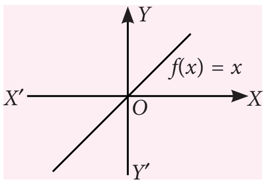


Graph of $f(x) = \log_a x$, when $a < 1$

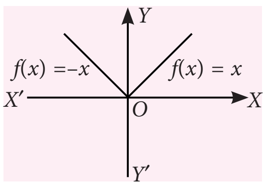
- **Explicit and implicit functions :** A function is said to be explicit if it can be expressed directly in terms of the independent variable otherwise is an implicit. *e.g.*, $y = \sin^{-1} x + \log x$ is explicit function, while $x^2 + y^2 = xy$ and $x^3y^2 = (a - x)^2 (b - y)^2$ are implicit functions.
- **Constant function :** Let k be a fixed real number. Then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function. The domain of the constant function $f(x) = k$ is the complete set of real numbers and the range of f is the singleton set $\{k\}$. The graph of a constant function is a straight line parallel to x-axis as shown in figure and it is above or below the x-axis according as k is positive or negative. If $k = 0$, then the straight line coincides with x-axis.



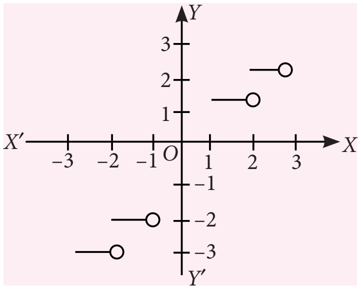
- **Identity function :** The function defined by $f(x) = x$ for all $x \in R$, is called the identity function on R. Clearly, the domain and range of the identity function is R. The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with positive direction of x-axis.



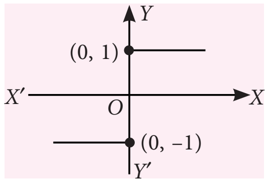
- **Modulus function :** The function defined by $f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$ is called the modulus function. The domain of the modulus function is the set R of all real numbers and the range is the set of all non-negative real numbers.



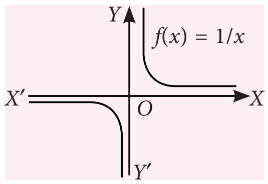
- **Greatest integer function :** Let $f(x) = [x]$ where $[x]$ denotes the greatest integer less than or equal to x. The domain is R and the range is I. *e.g.* $[1.1] = 1$, $[2.2] = 2$, $[-0.9] = -1$, $[-2.1] = -3$ etc. The function f defined by $f(x) = [x]$ for all $x \in R$, is called the greatest integer function.



- **Signum function :** The function defined by
$$f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$
or
$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$
is called the signum function. The domain is R and the range is the set $\{-1, 0, 1\}$.



- **Reciprocal function :** The function that associates each non-zero real number x to be reciprocal $\frac{1}{x}$ is called the reciprocal function. The domain and range of the reciprocal function are both equal to $R - \{0\}$ *i.e.*, the set of all non-zero real numbers. The graph is as shown.



- **Power function :** A function $f : R \to R$ defined by, $f(x) = x^\alpha$, $\alpha \in R$ is called a power function.

EVEN AND ODD FUNCTION

- **Even function :** If $f(-x) = f(x)$, $\forall\ x \in$ domain then $f(x)$ is called even function. *e.g.*, $f(x) = e^x + e^{-x}$, $f(x) = x^2$, $f(x) = x \sin x$, $f(x) = \cos x$, $f(x) = x^2 \cos x$, all are even functions.
- **Odd function :** If we put $(-x)$ in place of x in the given function and if $f(-x) = -f(x)$, $\forall\ x \in$ domain then $f(x)$ is called odd function. *e.g.*, $f(x) = e^x - e^x$, $f(x) = \sin x$, $f(x) = x^3$, $f(x) = x\cos x$, $f(x) = x^2 \sin x$, all are odd functions.
- **Properties of even and odd function**
  (i) The graph of even function is always symmetric with respect to y-axis. The graph of odd function is always symmetric with respect to origin.
  (ii) The product of two even functions is an even function.

(iii) The sum and difference of two even functions is an even function.

(iv) The sum and difference of two odd functions is an odd function.

(v) The product of two odd functions is an even function.

(vi) The product of an even and an odd function is an odd function. It is not essential that every function is even or odd. It is possible to have some functions which are neither even nor odd function. *e.g.*, $f(x) = x^2 + x^3$, $f(x) = \log_e x$, $f(x) = e^x$.

(vii) The sum of even and odd function is neither even nor odd function.

(viii) Zero function $f(x) = 0$ is the only function which is even and odd both.

PERIODIC FUNCTION

A function is said to be periodic function if its each value is repeated after a definite interval. So a function $f(x)$ will be periodic if a positive real number T exist such that, $f(x + T) = f(x)$, $\forall\ x \in$ domain. Here the least positive value of T is called the period of the function.

COMPOSITE FUNCTION

If $f : A \to B$ and $g : B \to C$ are two functions then the composite function of f and g, $gof : A \to C$ will be defined as $gof(x) = g(f(x))$, $\forall\ x \in A$

- **Properties of composition of function**
  (i) f is even, g is even $\Rightarrow fog$ is even function.
  (ii) f is odd, g is odd $\Rightarrow fog$ is odd function.
  (iii) f is even, g is odd $\Rightarrow fog$ is even function.
  (iv) f is odd, g is even $\Rightarrow fog$ is even function.
  (v) Composite of functions is not commutative *i.e.*, $fog \neq gof$.
  (vi) Composite of functions is associative *i.e.*, $(fog)oh = fo(goh)$
  (vii) Function gof will exist only when range of f is the subset of domain of g.
  (viii) If both f and g are one-one, or onto then fog and gof are also one-one or onto.

INVERSE FUNCTION

If $f : A \to B$ be a one-one onto (bijection) function, then the mapping $f^{-1} : B \to A$ which associates each element $b \in B$ with element $a \in A$, such that $f(a) = b$, is called the inverse function of the function $f : A \to B$.

$f^{-1} : B \to A, f^{-1}(b) = a \Rightarrow f(a) = b$

In terms of ordered pairs inverse function is defined as $f^{-1} = (b, a)$ if $(a, b) \in f$.

For the existence of inverse function, it should be one-one and onto.

- **Properties of Inverse function**
  (i) Inverse of a bijection is also a bijection and is unique.
  (ii) $(f^{-1})^{-1} = f$
  (iii) If $f : A \to B$ is bijection and $g : B \to A$ is inverse of f. Then $fog = I_B$ and $gof = I_A$. where, I_A and I_B are identity functions on the sets A and B respectively.
  (v) If $f : A \to B$ and $g : B \to C$ are two bijections, then $gof : A \to C$ is bijection and $(gof)^{-1} = (f^{-1}og^{-1})$.
  (iv) $fog \neq gof$ but if, $fog = gof$ then either $f^{-1} = g$ or $g^{-1} = f$ also, $(fog)(x) = (gof)(x) = (x)$.

PROBLEMS

Single Correct Answer Type

1. If $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$, then $f\left(\dfrac{2x}{1+x^2}\right)$ is equal to
(a) $[f(x)]^2$ (b) $[f(x)]^3$ (c) $2f(x)$ (d) $3f(x)$

2. If $f(x) = \dfrac{x-3}{x+1}$, then $f[f\{f(x)\}]$ equals
(a) x (b) $-x$ (c) $\dfrac{x}{2}$ (d) $-\dfrac{1}{x}$

3. If $f(x) = \cos(\log x)$, then the value of
$$f(x)\cdot f(4) - \frac{1}{2}\left[f\left(\frac{x}{4}\right) + f(4x)\right]$$
(a) 1 (b) -1 (c) 0 (d) ± 1

4. Let $f : R \to R$ be defined by $f(x) = 2x + |x|$, then $f(2x) + f(-x) - f(x) =$
(a) $2x$ (b) $2|x|$ (c) $-2x$ (d) $-2|x|$

5. If $f(x + ay, x - ay) = axy$, then $f(x, y)$ is equal to
(a) xy (b) $x^2 - a^2y^2$
(c) $\dfrac{x^2 - y^2}{4}$ (d) $\dfrac{x^2 - y^2}{a^2}$

6. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, then
(a) $f\left(\dfrac{\pi}{4}\right) = 2$ (b) $f(-\pi) = 2$
(c) $f(\pi) = 1$ (d) $f\left(\dfrac{\pi}{2}\right) = -1$

7. If $e^{f(x)} = \dfrac{10+x}{10-x}$, $x \in (-10, 10)$ and

$f(x)=kf\left(\frac{200x}{100+x^2}\right)$,then $k =$

(a) 0.5 (b) 0.6 (c) 0.7 (d) 0.8

8. The graph of the function $y = f(x)$ is symmetrical about the line $x = 2$, then

(a) $f(x) = -f(-x)$ (b) $f(2 + x) = f(2 - x)$
(c) $f(x) = f(-x)$ (d) $f(x + 2) = f(x - 2)$

9. Mapping $f : R \rightarrow R$ which is defined as $f(x) = \cos x$, $x \in R$ will be

(a) Neither one-one nor onto
(b) One-one
(c) Onto
(d) One-one onto

10. Let $f : N \rightarrow N$ defined by $f(x) = x^2 + x + 1, x \in N$, then f is

(a) One-one onto
(b) Many one onto
(c) One-one but not onto
(d) None of these

11. Set A has 3 elements and set B has 4 elements. The number of injection that can be defined from A to B is

(a) 144 (b) 12 (c) 24 (d) 64

12. Let $f : R \rightarrow R$ be a function defined by $f(x)=\frac{x-m}{x-n}$, where $m \neq n$. Then

(a) f is one-one onto (b) f is one-one into
(c) f is many one onto (d) f is many one into

13. Which one of the following is a bijective function on the set of real numbers

(a) $2x - 5$ (b) $|x|$ (c) x^2 (d) $x^2 + 1$

14. Let the function $f : R \rightarrow R$ be defined by $f(x) = 2x + \sin x, x \in R$. Then f is

(a) One-to-one and onto
(b) One-to-one but not onto
(c) Onto but not one-to-one
(d) Neither one-to-one nor onto

15. If $f : [0, \infty) \rightarrow [0, \infty)$ and $f(x)=\frac{x}{1+x}$, then f is

(a) One-one and onto
(b) One-one but not onto
(c) Onto but not one-one
(d) Neither one-one nor onto

16. If $f : R \rightarrow S$ defined by $f(x)=\sin x-\sqrt{3}\cos x+1$ is onto, then the interval of S is

(a) $[-1, 3]$ (b) $[1, 1]$ (c) $[0, 1]$ (d) $[0, -1]$

17. If R denotes the set of all real numbers then the function $f : R \rightarrow R$ defined $f(x) = [x]$

(a) One-one only
(b) Onto only
(c) Both one-one and onto
(d) Neither one-one nor onto

18. $f(x)=x+\sqrt{x^2}$ is a function from $R \rightarrow R$, then $f(x)$ is

(a) Injective (b) Surjective
(c) Bijective (d) None of these

19. The period of $f(x) = x - [x]$, if it is periodic, is

(a) $f(x)$ is not periodic (b) $\frac{1}{2}$
(c) 1 (d) 2

20. If $f(x) = ax + b$ and $g(x) = cx + d$, then $f(g(x)) = g(f(x))$ is equivalent to

(a) $f(a) = g(c)$ (b) $f(b) = g(b)$
(c) $f(d) = g(b)$ (d) $f(c) = g(a)$

21. The domain of the function $f(x)=\frac{\sin^{-1}(3-x)}{\log(|x|-2)}$ is

(a) $[2, 4]$ (b) $(2, 3) \cup (3, 4]$
(c) $[2, \infty)$ (d) $(-\infty, -3) \cup [2, \infty)$

22. The function $f(x)=\frac{\sec^{-1}x}{\sqrt{x-[x]}}$, where [.] denotes the greatest integer less than or equal to x is defined for all x belonging to

(a) R
(b) $R - \{(-1, 1) \cup (n \mid n \in Z)\}$
(c) $R^+ - (0, 1)$
(d) $R^+ - \{n \mid n \in N\}$

23. Domain of the function $f(x)=\left[\log_{10}\left(\frac{5x-x^2}{4}\right)\right]^{1/2}$ is

(a) $-\infty < x < \infty$ (b) $1 \le x \le 4$
(c) $4 \le x \le 16$ (d) $-1 \le x \le 1$

24. $f : [0, 1] \rightarrow R$ is a differentiable function such that $f(0) = 0$ and $|f'(x)| \le k\,|f(x)|$ for all $x \in [0, 1]$, $(k > 0)$, then which of the following is/are always true ?

(a) $f(x) = 0, \forall\, x \in R$
(b) $f(x) = 0, \forall\, x \in [0, 1]$
(c) $f(x) \neq 0, \forall\, x \in [0, 1]$
(d) $f(1) = k$

25. Domain of definition of the function

$$f(x)=\frac{3}{4-x^2}+\log_{10}(x^3-x)\text{ , is}$$

(a) $(1, 2)$
(b) $(-1, 0) \cup (1, 2)$
(c) $(1, 2) \cup (2, \infty)$
(d) $(-1, 0) \cup (1, 2) \cup (2, \infty)$

26. The domain of the function
$f(x)=\sqrt{x-x^2}+\sqrt{4+x}+\sqrt{4-x}$ is
(a) $[-4, \infty)$ (b) $[-4, 4]$ (c) $[0, 4]$ (d) $[0, 1]$

27. The domain of the function $\sqrt{\log(x^2-6x+6)}$ is
(a) $(-\infty, \infty)$
(b) $(-\infty, 3-\sqrt{3}) \cup (3+\sqrt{3}, \infty)$
(c) $(-\infty, 1] \cup [5, \infty)$ (d) $[0, \infty)$

28. The natural domain of the real valued function defined by $f(x)=\sqrt{x^2-1}+\sqrt{x^2+1}$ is
(a) $1 < x < \infty$ (b) $-\infty < x < \infty$
(c) $-\infty < x < -1$ (d) $(-\infty, \infty) - (-1, 1)$

29. The domain of the function $f(x)=\dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is
(a) $[1, 2)$ (b) $[2, 3)$ (c) $[1, 2]$ (d) $[2, 3]$

30. The range of $f(x)=\sec\left(\dfrac{\pi}{4}\cos^2 x\right), -\infty < x < \infty$ is
(a) $[1, \sqrt{2}]$ (b) $[1, \infty)$
(c) $[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$ (d) $(-\infty, -1] \cup [1, \infty)$

31. If $f(x) = a\cos(bx + c) + d$, then range of $f(x)$ is
(a) $[d + a, d + 2a]$ (b) $[a - d, a + d]$
(c) $[d + a, a - d]$ (d) $[d - a, d + a]$

32. Range of $f(x)=\dfrac{x^2+34x-71}{x^2+2x-7}$ is
(a) $[5, 9]$ (b) $(-\infty, 5] \cup [9, \infty)$
(c) $(5, 9)$ (d) None of these

33. The function $f : R \to R$ is defined by $f(x) = \cos^2 x + \sin^4 x$ for $x \in R$, then $f(R) =$
(a) $\left(\frac{3}{4}, 1\right]$ (b) $\left[\frac{3}{4}, 1\right)$ (c) $\left[\frac{3}{4}, 1\right]$ (d) $\left(\frac{3}{4}, 1\right)$

34. If x is real, then value of the expression $\dfrac{x^2+14x+9}{x^2+2x+3}$ lies between
(a) 5 and 4 (b) 5 and –4
(c) – 5 and 4 (d) None of these

35. The function $f(x)=\sin\left(\log(x+\sqrt{x^2+1})\right)$ is
(a) Even function
(b) Odd function
(c) Neither even nor odd
(d) Periodic function

36. The inverse of the function $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}+2$ is given by
(a) $\log_e\left(\dfrac{x-2}{x-1}\right)^{1/2}$ (b) $\log_e\left(\dfrac{x-1}{3-x}\right)^{1/2}$
(c) $\log_e\left(\dfrac{x}{2-x}\right)^{1/2}$ (d) $\log_e\left(\dfrac{x-1}{x+1}\right)^{-2}$

37. If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is
(a) $\left(\dfrac{1}{2}\right)^{x(x-1)}$ (b) $\dfrac{1}{2}(1+\sqrt{1+4\log_2 x})$
(c) $\dfrac{1}{2}(1-\sqrt{1+4\log_2 x})$ (d) Not defined

38. If $f(x)=\dfrac{2x-1}{x+5}(x \neq -5)$, then $f^{-1}(x)$ is equal to
(a) $\dfrac{x+5}{2x-1}, x \neq \dfrac{1}{2}$ (b) $\dfrac{5x+1}{2-x}, x \neq 2$
(c) $\dfrac{5x-1}{2-x}, x \neq 2$ (d) $\dfrac{x-5}{2x+1}, x \neq \dfrac{1}{2}$

Multiple Correct Answer Type

39. Let $f(x)$ be a non constant polynomial satisfying the relation $f(x) \cdot f(y) = f(x) + f(y) + f(xy) - 2$ for all real x and y and $f(0) \neq 1$, suppose $f(4) = 65$. Then
(a) $f^1(x)$ is a polynomial of degree 2
(b) roots of $f^1(x) = 2x + 1$ are real
(c) $xf^1(x) = 3[f(x) - 1]$
(d) $f^1(-1) = 3$

40. If a function satisfies $(x - y) f(x + y) - (x + y) f(x - y) = 2(x^2y - y^3)\ \forall\, x, y \in R$ and $f(1) = 2$, then
(a) $f(x)$ must be polynomial function
(b) $f(3) = 12$
(c) $f(0) = 0$
(d) $f(x)$ may not be differentiable

41. Solutions of the equations $[x] + [y] = [x]\,[y]$ is/are where [.] denotes the greater integer function
(a) $2 \le x < 3$ and $2 \le y < 3$
(b) $0 \le x < 1$ and $0 \le y < 1$
(c) $0 < x \le 2$ and $0 < y \le 2$
(d) None

42. Let $f(x) = a_1\cos(\alpha_1 + x) + a_2\cos(\alpha_2 + x) + \ldots + a_n\cos(\alpha_n + x)$.
If $f(x)$ vanishes for $x = 0$ and $x = x_1$ (where $x_1 \ne k\pi$, $k \in Z$), then
(a) $a_1\cos\alpha_1 + a_2\cos\alpha_2 + \ldots + a_n\cos\alpha_n = 0$
(b) $a_1\sin\alpha_1 + a_2\sin\alpha_2 + \ldots + a_n\sin\alpha_n = 0$
(c) $f(x) = 0$ has only two solutions $0, x_1$
(d) $f(x)$ is identically zero $\forall\, x$

43. Consider the real valued function satisfying $2f(\sin x) + f(\cos x) = x$, then
(a) domain of $f(x)$ is R
(b) domain of $f(x)$ is $[-1, 1]$
(c) range of $f(x)$ is $\left[\frac{-2\pi}{3}, \frac{\pi}{6}\right]$
(d) range of $f(x)$ is $\left[\frac{-2\pi}{3}, \frac{\pi}{3}\right]$

44. Let $f(x)=\begin{cases} x^2-4x+3 & x<3 \\ x-4 & x\ge 3\end{cases}$ and

$g(x)=\begin{cases} x-3 & x<4 \\ (x+1)^2+1 & x\ge 4\end{cases}$ then

(a) $(f-g)\left(\frac{7}{2}\right)=-1$ (b) $fog(3) = 3$
(c) $(fg)(2) = 1$
(d) $(f+g)\left(\frac{7}{2}\right)-(f-g)(4)=26$

Comprehension Type

Paragraph for Q. No. 45 and 46

Let $f : [2, \infty) \to [1, \infty)$ defined by $f(x) = 2^{x^4 - 4x^2}$ and $g:\left[\frac{\pi}{2},\pi\right]\to A$ defined by $g(x)=\frac{\sin x+4}{\sin x-2}$ be two invertible function, then

45. $f^{-1}(x)$ is equal to
(a) $\sqrt{2+\sqrt{4-\log_2 x}}$ (b) $\sqrt{2+\sqrt{4+\log_2 x}}$
(c) $\sqrt{2-\sqrt{4+\log_2 x}}$ (d) None of these

46. The set A is equal to
(a) $[-5, -2]$ (b) $[2, 5]$
(c) $[-5, 2]$ (d) $[-3, -2]$

Matrix-Match Type

47. Match the following.

	Column I		Column II
(A)	The interval containing the complete set of solution of the equation $\left\lvert\frac{1-x^2}{x}\right\rvert+\lvert x\rvert=\left\lvert\frac{1}{x}\right\rvert$ are	(p)	$\left[0,\frac{1}{2}\right]$
(B)	The interval containing the complete set of values of 'a', for which $(a + 1)\,x + ay - 1 = 0$ is a normal to the curve $xy = 1$, are	(q)	$[-1, 1] - \{0\}$
(C)	Complete set of values of 'a' for which equation $a\sin^2 x + \lvert\cos x\rvert - 2a = 0$ has atleast one solution belongs to the interval	(r)	$[0, 1]$
(D)	The interval containing the range of the function $\frac{1}{1+2\cos^2 x+3\cos^4 x+4\cos^6 x+\ldots\infty}$	(s)	$[-1, 0]$
		(t)	$\left[-1,\frac{1}{2}\right]$

Integer Answer Type

48. Find the natural number c for which $\sum_{r=1}^{n} f(c+r)=16(2^n-1)$ where the function satisfies the relation $f(x + y) = f(x)\cdot f(y)$ for all $x, y \in N$ (natural numbers) and $f(1) = 2$.

49. For non negative integers m, n define a function as follows

$$f(m,n)=\begin{cases} n+1 & \text{if} \quad m=0 \\ f(m-1,1) & \text{if} \quad m\ne 0, n=0 \\ f(m-1, f(m,n-1)) & \text{if} \quad m\ne 0, n\ne 0\end{cases}$$

Then the value of $f(1, 1)$ is

50. If function f satisfies the relation $f(x)\cdot f'(-x)=f(-x).f'(x)$ for all x and $f(0)=3$, now if $f(3)=3$, then the value of $f(-3)$ is

51. Number of real values of x, satisfying the equation $[x]^2-5[x]+6-\sin x=0$, $[\cdot]$ denoting the greatest integer function is

52. If $f(x+2)-5f(x+1)+6f(x)=0$ for all, and $f(x)=Aa_1^x+Ba_2^x$, then a_1+a_2 is

53. Let f be a function from the set of positive integers to the set of real numbers *i.e.*, $f:N\to R$ such that
(i) $f(1)=1$
(ii) $f(1)+2f(2)+3f(3)+...+nf(n)=n(n+1)f(n)$; for $n\geq 2$.
Find the value of $800\cdot f(200)$.

54. If for all real values of u and v, $2f(u)\cos v=f(u-v)+f(u+v)$ prove that for all real values of x,
(i) $f(x)+f(-x)=2a\cos x$
(ii) $f(\pi-x)+f(-x)=0$
(iii) $f(\pi-x)+f(x)=2b\sin x$
Then $f(x)=a\cos mx+b\sin nx$ where a and b are arbitrary constants, find $m+n$.

55. Let $f(x+y+1)=\left(\sqrt{f(x)}+\sqrt{f(y)}\right)^2$ for all $x,y\in R$ and $f(0)=1$. Then $f(x)=(x+1)^m$. Find m.

SOLUTIONS

1. (c) : Given, $f(x)=\log\left(\frac{1+x}{1-x}\right)$

$$\therefore\quad f\left(\frac{2x}{1+x^2}\right)=\log\left[\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right]=\log\left[\frac{x^2+1+2x}{x^2+1-2x}\right]$$

$$=\log\left[\frac{1+x}{1-x}\right]^2=2\log\left[\frac{1+x}{1-x}\right]=2f(x)$$

2. (a) : $f[f(x)]=\dfrac{f(x)-3}{f(x)+1}=\dfrac{\left(\frac{x-3}{x+1}\right)-3}{\left(\frac{x-3}{x+1}\right)+1}$

$$=\frac{x-3-3x-3}{x-3+x+1}=\frac{3+x}{1-x}$$

Now, $f[f(f(x))]=f\left(\dfrac{3+x}{1-x}\right)=\dfrac{\left(\frac{3+x}{1-x}\right)-3}{\left(\frac{3+x}{1-x}\right)+1}$

$$=\frac{3+x-3+3x}{3+x+1-x}=x$$

3. (c) : $f(x)=\cos(\log x)$

Now let $y=f(x)\cdot f(4)-\dfrac{1}{2}\left[f\left(\dfrac{x}{4}\right)+f(4x)\right]$

$$\Rightarrow\quad y=\cos(\log x)\cdot\cos(\log 4)-\frac{1}{2}\left[\cos\left(\log\left(\frac{x}{4}\right)\right)+\cos(\log 4x)\right]$$

$$\Rightarrow\quad y=\cos(\log x)\cos(\log 4)-\frac{1}{2}[\cos(\log x-\log 4)+\cos(\log x+\log 4)]$$

$$\Rightarrow\quad y=\cos(\log x)\cos(\log 4)-\frac{1}{2}[2\cos(\log x)\cos(\log 4)]$$

$\Rightarrow\quad y=0.$

4. (b) : $f(2x)=2(2x)+|2x|=4x+2|x|$,
$f(-x)=-2x+|-x|=-2x+|x|$, $f(x)=2x+|x|$
$\Rightarrow f(2x)+f(-x)-f(x)=4x+2|x|+|x|-2x-2x-|x|=2|x|$.

5. (c) : Given $f(x+ay,x-ay)=axy$
Let $x+ay=u$ and $x-ay=v$ ...(i)

Then $x=\dfrac{u+v}{2}$ and $y=\dfrac{u-v}{2a}$

Substituting the value of x and y in (i), we obtain

$$f(u,v)=\frac{u^2-v^2}{4}\Rightarrow f(x,y)=\frac{x^2-y^2}{4}.$$

6. (d) : $f(x)=\cos[\pi^2]x+\cos[-\pi^2]x$
$f(x)=\cos(9x)+\cos(-10x)=\cos(9x)+\cos(10x)$

$$=2\cos\left(\frac{19x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$f\left(\frac{\pi}{2}\right)=2\cos\left(\frac{19\pi}{4}\right)\cos\left(\frac{\pi}{4}\right);$$

$$f\left(\frac{\pi}{2}\right)=2\times\frac{-1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}=-1.$$

7. (a) : $e^{f(x)}=\dfrac{10+x}{10-x}$, $x\in(-10,10)$

$$\Rightarrow\quad f(x)=\log\left(\frac{10+x}{10-x}\right)$$

$$\Rightarrow\quad f\left(\frac{200x}{100+x^2}\right)=\log\left[\frac{10+\frac{200x}{100+x^2}}{10-\frac{200x}{100+x^2}}\right]$$

$$=\log\left[\frac{10(10+x)}{10(10-x)}\right]^2=2\log\left(\frac{10+x}{10-x}\right)=2f(x)$$

$$\therefore\quad f(x)=\frac{1}{2}f\left(\frac{200x}{100+x^2}\right)\Rightarrow k=\frac{1}{2}=0.5.$$

8. (b) : $f(x) = f(-x) \Rightarrow f(0 + x) = f(0 - x)$ is symmetrical about $x = 0$.

$\therefore$ $f(2 + x) = f(2 - x)$ is symmetrical about $x = 2$.

9. (a) : Let $x_1, x_2 \in R$, then $f(x_1) = \cos x_1$, $f(x_2) = \cos x_2$, so $f(x_1) = f(x_2)$

$\Rightarrow$ $\cos x_1 = \cos x_2 \Rightarrow x_1 = 2n\pi \pm x_2$

$\Rightarrow$ $x_1 \neq x_2$, so it is not one-one.

Again the value of f-image of x lies in between –1 to 1

$\Rightarrow$ $f(R) = \{f(x) : -1 \le f(x) \le 1\}$

So other numbers of co-domain is not f-image. $f(R) \in R$, so it is also not onto. So this mapping is neither one-one nor onto.

10. (c) : Let $x, y \in N$ such that $f(x) = f(y)$

Then $f(x) = f(y) \Rightarrow x^2 + x + 1 = y^2 + y + 1$

$\Rightarrow$ $(x - y)(x + y + 1) = 0$

$\Rightarrow$ $x = y$ or $x = -(y + 1) \notin N$ (Rejected)

$\therefore$ f is one-one.

Now, Let $f(x) = 1 \Rightarrow x^2 + x + 1 = 1$

$\Rightarrow$ $x(x + 1) = 0$

$\Rightarrow$ $x = 0$ or $x = -1$ (Not possible)

$\therefore$ f is not onto.

11. (c) : The total number of injective functions from a set A containing 3 elements to a set B containing 4 elements is equal to the total number of arrangements of 4 by taking 3 at a time *i.e.*, ${}^4P_3 = 24$.

12. (b) : For any $x, y \in R$, we have

$$f(x) = f(y) \Rightarrow \frac{x-m}{x-n} = \frac{y-m}{y-n} \Rightarrow x = y$$

$\therefore$ f is one-one.

Let $\alpha \in R$ such that $f(x) = \alpha \Rightarrow \dfrac{x-m}{x-n} = \alpha$

$$\Rightarrow \quad x = \frac{m-n\alpha}{1-\alpha}$$

Clearly $x \notin R$ for $\alpha = 1$. So, f is not onto.

13. (a) : $|x|$, x^2 and $x^2 + 1$ is not one-one. But $2x - 5$ is one-one as $f(x) = f(y) \Rightarrow 2x - 5 = 2y - 5 \Rightarrow x = y$

Also, $f(x) = 2x - 5$ is onto. $\therefore$ $f(x) = 2x - 5$ is bijective.

14. (a) : $f'(x) = 2 + \cos x > 0$. So, $f(x)$ is strictly monotonic increasing so, $f(x)$ is one-to-one and onto.

15. (b) : $f'(x) = \dfrac{1}{(1+x)^2} > 0$, $\forall\ x \in [0\ \infty)$ and range $\in [0, 1)$

$\Rightarrow$ Function is one-one but not onto.

16. (a) : $-\sqrt{1+(-\sqrt{3})^2} \le (\sin x - \sqrt{3}\cos x) \le \sqrt{1+(-\sqrt{3})^2}$

$-2 \le (\sin x - \sqrt{3}\cos x) \le 2$

$-2 + 1 \le (\sin x - \sqrt{3}\cos x + 1) \le 2 + 1$

$-1 \le (\sin x - \sqrt{3}\cos x + 1) \le 3$ *i.e.*, range = $[-1, 3]$

$\therefore$ For f to be onto $S = [-1, 3]$.

17. (d) : Let $f(x_1) = f(x_2) \Rightarrow [x_1] = [x_2] \neq x_1 = x_2$

{For example, if $x_1 = 1{\cdot}4$, $x_2 = 1{\cdot}5$, then $[1{\cdot}4] = [1{\cdot}5] = 1$}

$\therefore$ f is not one-one.

Also, f is not onto as its range I (set of integers) is a proper subset of its co-domain R.

18. (d) : We have $f(x) = x + \sqrt{x^2} = x + |x|$

Clearly f is not one-one as $f(-1) = f(-2) = 0$ but $-1 \neq 2$.

Also f is not onto as $f(x) \ge 0\ \forall\ x \in R$,

Also range of $f = [0, \infty) \subset R$.

19. (c) : Let $f(x)$ be periodic with period T.

Then, $f(x + T) = f(x)$ for all $x \in R$

$\Rightarrow$ $x + T - [x + T] = x - [x]$, for all $x \in R$

$\Rightarrow$ $x + T - x = [x + T] - [x]$

$\Rightarrow$ $[x + T] - [x] = T$ for all $x \in R \Rightarrow T = 1, 2, 3, 4,...$

The smallest value of T satisfying $f(x + T) = f(x)$ for all $x \in R$ is 1.

Hence $f(x) = x - [x]$ has period 1.

20. (c) : We have $f(x) = ax + b$, $g(x) = cx + d$ and $f(g(x)) = g(f(x))$

$\Rightarrow$ $f(cx + d) = g(ax + b)$

$\Rightarrow$ $a(cx + d) + b = c(ax + b) + d$

$\Rightarrow$ $ad + b = cb + d \Rightarrow f(d) = g(b)$.

21. (b) : $f(x) = \dfrac{\sin^{-1}(3-x)}{\log(|x|-2)}$

Let $g(x) = \sin^{-1}(3 - x) \Rightarrow -1 \le 3 - x \le 1$

Domain of $g(x)$ is $[2, 4]$

and let $h(x) = \log(|x| - 2) \Rightarrow |x| - 2 > 0$

$\Rightarrow$ $|x| > 2 \Rightarrow x < -2$ or $x > 2 \Rightarrow (-\infty, -2) \cup (2, \infty)$

We know that

$$(f/g)(x) = \frac{f(x)}{g(x)} \forall x \in D_1 \cap D_2 - \{x \in R : g(x) = 0\}$$

$\therefore$ Domain of $f(x) = (2, 4] - \{3\} = (2, 3) \cup (3, 4]$.

22. (b) : The function $\sec^{-1} x$ is defined for all $x \in R - (-1, 1)$ and the function $\dfrac{1}{\sqrt{x-[x]}}$ is defined for all $x \in R - Z$. So the given function is defined for all $x \in R - \{(-1, 1) \cup (n \mid n \in Z)\}$.

23. (b) : We have $f(x) = \left[\log_{10}\left(\dfrac{5x-x^2}{4}\right)\right]^{1/2}$ ...(i)

From (i), clearly $f(x)$ is defined for those values of x for which $\log_{10}\left[\dfrac{5x-x^2}{4}\right] \ge 0$

$\Rightarrow \left(\dfrac{5x-x^2}{4}\right) \ge 10^0 \Rightarrow \left(\dfrac{5x-x^2}{4}\right) \ge 1$

$\Rightarrow x^2 - 5x + 4 \le 0 \Rightarrow (x-1)(x-4) \le 0$

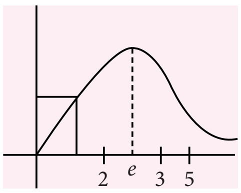
Hence domain of the function is [1, 4].

24. (b) : $(f'(x))^2 - k^2(f(x))^2 \le 0$

$\Rightarrow (f'(x) - kf(x))$
$(f'(x) + kf(x)) \le 0$

$\Rightarrow (f(x)e^{-kx})' \, (f(x)e^{kx})' \le 0$

$\Rightarrow$ Exactly one of the functions $g_1(x) = f(x)e^{-kx}$



or $g_2(x) = f(x)e^{kx}$ is non-decreasing.

But $f(0) = 0 \Rightarrow$ both function g_1 and g_2 have a value zero at $x = 0\ \forall\ x \in [0, 1]$, $g_1(0) = 0$ and g_1 increasing $\Rightarrow g_1(x) \ge 0 \Rightarrow f(x) \ge 0$

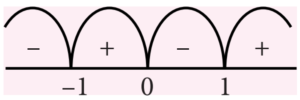
$g_2(0) = 0$ and g_2 decreasing $\Rightarrow g_2(x) \le 0 \Rightarrow f(x) \le 0$

$\Rightarrow f(x) = 0\ \forall\ x \in [0, 1]$

25. (d) : $f(x) = \dfrac{3}{4-x^2} + \log_{10}(x^3 - x)$.

So, $4 - x^2 \ne 0 \Rightarrow x \ne \pm\sqrt{4}$

and $x^3 - x > 0 \Rightarrow x(x^2 - 1) > 0$



$\therefore$ $D = (-1, 0) \cup (1, \infty) - \{\sqrt{4}\}$ *i.e.*,
$D = (-1, 0) \cup (1, 2) \cup (2, \infty)$.

26. (d) : $f(x) = \sqrt{x-x^2} + \sqrt{4+x} + \sqrt{4-x}$

Clearly $f(x)$ is defined, if $4 + x \ge 0 \Rightarrow x \ge -4$

$4 - x \ge 0 \Rightarrow x \le 4$

$x(1 - x) \ge 0 \Rightarrow x \ge 0$ and $x \le 1$

$\therefore$ Domain of $f = (-\infty, 4] \cap [-4, \infty) \cap [0, 1] = [0, 1]$.

27. (c) : $\log(x^2 - 6x + 6) \ge 0$

i.e., $x^2 - 6x + 6 \ge 1$

$x^2 - 6x + 5 \ge 0$

$(x - 1)(x - 5) \ge 0$ *i.e.*, $x \in (-\infty, 1] \cup [5, \infty)$

28. (d) : $f(x) = \sqrt{x^2-1} + \sqrt{x^2+1} \Rightarrow f(x) = y_1 + y_2$

Domain of $y_1 = \sqrt{x^2-1} \Rightarrow x^2 - 1 \ge 0 \Rightarrow x^2 \ge 1$

$x \in (-\infty, \infty) - (-1, 1)$ and Domain of y_2 is real number,

$\therefore$ Domain of $f(x) = (-\infty, \infty) - (-1, 1)$.

29. (b) : To define $f(x)$, $9 - x^2 > 0 \Rightarrow -3 < x < 3$ ...(i)

$-1 \le (x - 3) \le 1 \Rightarrow 2 \le x \le 4$ ...(ii)

From (i) and (ii), $2 \le x < 3$ *i.e.*, [2, 3).

30. (a) : $f(x) = \sec\left(\dfrac{\pi}{4}\cos^2 x\right)$

We know that, $0 \le \cos^2 x \le 1$ Also, at $\cos x = 0$, $f(x) = 1$ and at $\cos x = 1$, $f(x) = \sqrt{2}$ $\therefore$ $1 \le x \le \sqrt{2} \Rightarrow x \in [1, \sqrt{2}]$.

31. (d) : $f(x) = a\cos(bx + c) + d$ ...(i)

For minimum $\cos(bx + c) = -1$

from (i), $f(x) = -a + d = (d - a)$

For maximum $\cos(bx + c) = 1$

from (i), $f(x) = a + d = (d + a)$

$\therefore$ Range of $f(x) = [d - a, d + a]$.

32. (b) : Let $\dfrac{x^2+34x-71}{x^2+2x-7} = y$

$\Rightarrow x^2(1 - y) + 2(17 - y)x + (7y - 71) = 0$

For real value of x, discriminant, $D \ge 0$

$\Rightarrow y^2 - 14y + 45 \ge 0 \Rightarrow y \ge 9, y \le 5$.

33. (c) : $y = f(x) = \cos^2 x + \sin^4 x$

$\Rightarrow y = f(x) = \cos^2 x + \sin^2 x(1 - \cos^2 x)$

$\Rightarrow y = \cos^2 x + \sin^2 x - \sin^2 x \cos^2 x$

$\Rightarrow y = 1 - \sin^2 x \cos^2 x \Rightarrow y = 1 - \dfrac{1}{4} \cdot \sin^2 2x$

$\therefore \dfrac{3}{4} \le f(x) \le 1$, $(\because 0 \le \sin^2 2x \le 1)$

$\Rightarrow f(R) \in [3/4, 1]$.

34. (c) : $\dfrac{x^2+14x+9}{x^2+2x+3} = y$

$\Rightarrow x^2(y - 1) + 2x(y - 7) + (3y - 9) = 0$

Since x is real, $\therefore 4(y - 7)^2 - 4(3y - 9)(y - 1) \ge 0$

$\Rightarrow 4(y^2 + 49 - 14y) - 4(3y^2 + 9 - 12y) \ge 0$

$\Rightarrow 4y^2 + 196 - 56y - 12y^2 - 36 + 48y \ge 0$

$\Rightarrow 8y^2 + 8y - 160 \le 0 \Rightarrow y^2 + y - 20 \le 0$

$\Rightarrow (y + 5)(y - 4) \le 0$ $\therefore$ y lies between – 5 and 4.

35. (b) : $f(x) = \sin\left(\log(x + \sqrt{1+x^2})\right)$

$\Rightarrow f(-x) = \sin\left(\log(-x + \sqrt{1+x^2})\right)$

$\Rightarrow f(-x) = \sin\log\left((\sqrt{1+x^2} - x)\dfrac{(\sqrt{1+x^2}+x)}{(\sqrt{1+x^2}+x)}\right)$

$\Rightarrow f(-x) = \sin\log\left[\dfrac{1}{(x+\sqrt{1+x^2})}\right]$

$\Rightarrow\ f(-x) = \sin\left[\log(x+\sqrt{1+x^2})^{-1}\right]$

$\Rightarrow\ f(-x) = \sin\left[-\log(x+\sqrt{1+x^2})\right]$

$\Rightarrow\ f(-x) = -\sin\left[\log(x+\sqrt{1+x^2})\right] \Rightarrow f(-x) = -f(x)$

$\therefore\ f(x)$ is odd function.

36. (b) : $y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}+2 \Rightarrow y=\dfrac{e^{2x}-1}{e^{2x}+1}+2$

$\Rightarrow\ e^{2x}=\dfrac{1-y}{y-3}=\dfrac{y-1}{3-y} \Rightarrow x=\dfrac{1}{2}\log_e\left(\dfrac{y-1}{3-y}\right)$

$\Rightarrow\ f^{-1}(y)=\log_e\left(\dfrac{y-1}{3-y}\right)^{1/2} \Rightarrow f^{-1}(x)=\log_e\left(\dfrac{x-1}{3-x}\right)^{1/2}$

37. (b) : Given $f(x) = 2^{x(x-1)} \Rightarrow x(x-1) = \log_2 f(x)$

$\Rightarrow\ x^2 - x - \log_2 f(x) = 0 \Rightarrow x = \dfrac{1\pm\sqrt{1+4\log_2 f(x)}}{2}$

Only $x=\dfrac{1+\sqrt{1+4\log_2 f(x)}}{2}$ lies in the domain

$\therefore\ f^{-1}(x) = \dfrac{1}{2}[1+\sqrt{1+4\log_2 x}]$.

38. (b) : Let $f(x) = y \Rightarrow x = f^{-1}(y)$

Now, $y=\dfrac{2x-1}{x+5}, (x\neq -5)$

$xy + 5y = 2x - 1 \Rightarrow 5y + 1 = 2x - xy$

$\Rightarrow\ x(2-y) = 5y + 1 \Rightarrow x = \dfrac{5y+1}{2-y}$

$\Rightarrow\ f^{-1}(y)=\dfrac{5y+1}{2-y}$

$\therefore\ f^{-1}(x)=\dfrac{5x+1}{2-x},\ x\neq 2$

39. (a, b, c, d) : Given $f(x)\cdot f(y) = f(x) + f(y) + f(xy) - 2$ $\forall\ x, y \in R$.

Put $x = y = 1$ then $f(1)^2 - 3f(1) + 2 = 0$

$\Rightarrow\ f(1) = 1$ or $f(1) = 2$

If $f(1) = 1$, then $f(x) = 1\ \forall\ x \in R$.

A contradiction $\because$ degree of $f(x)$ is positive.

$\therefore\ f(1) \neq 1$. hence, $f(1) = 2$.

Replace 'y' with $\dfrac{1}{x}$ then

$f(x)\cdot f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)$ $\quad(\because f(1) = 2)$

$\therefore\ f(x)$ must be in the form $x^n + 1$ or $-x^n + 1$.

$\because\ f(4) = 65, f(x) = x^3 + 1 \Rightarrow f'(x) = 3x^2$.

40. (a, b, c) : $(x-y)f(x+y) - (x+y)f(x-y) = 2y((x-y)(x+y))$

Let $x - y = u;\ x + y = v$

$uf(v) - vf(u) = \dfrac{2uv(v-u)}{2}$

$\dfrac{f(v)}{v}-\dfrac{f(u)}{u} = v - u$

$\left(\dfrac{f(v)}{v}-v\right)=\left(\dfrac{f(u)}{u}-u\right)$ = constant

Let $\dfrac{f(x)}{x} - x = \lambda$

$\Rightarrow\ f(x) = (\lambda x + x^2)$

$f(1) = 2$

$\lambda + 1 = 2 \Rightarrow \lambda = 1 \quad f(x) = x^2 + x$

41. (a, b) : Let $a = [x] + [y] = [x]\cdot[y]$

Then from the given equation, we have $a + b = a\cdot b$

$\Rightarrow\ ab - a - b = 0$

$\Rightarrow\ ab - a - b + 1 = 1 \Rightarrow (a-1)(b-1) = 1$.

This is possible if (i) $a - 1 = 1, b - 1 = 1$ or (ii) $a - 1 = -1, b - 1 = -1$

Now, for (i), $a = 2$ and $b = 2$

And for (ii) $a = 0$ and $b = 0$

Thus $[x] = 2$ and $[y] = 2$ or $[x] = 0, [y] = 0$

But $[x] = 2 \Rightarrow 2 \le x < 3$ and $[y] = 2 \Rightarrow 2 \le y < 3$.

Again $[x] = 0 \Rightarrow 0 \le x < 1$ and $[y] = 0 \Rightarrow 0 \le y < 1$

Thus, the solution sets are

$0 \le x < 1$ and $0 \le y < 1$; $2 \le x < 3$ and $2 \le y < 3$.

42. (a, b, d) : $f(0) = 0 \Rightarrow a_1\cos\alpha_1 + a_2\cos\alpha_2 + ... + a_n\cos\alpha_n = 0$

$f(x_1) = 0 \Rightarrow (a_1\cos\alpha_1 + a_2\cos\alpha_2 + ... + a_n\cos\alpha_n)\cos x_1 - (a_1\sin\alpha_1 + a_2\sin\alpha_2 + ... + a_n\sin\alpha_n)\sin x_1 = 0$

$\Rightarrow\ a_1\sin\alpha_1 + a_2\sin\alpha_2 + ... + a_n\sin\alpha_n = 0$ $\quad(\because x_1 \neq n\pi)$

$\therefore\ a_1\cos\alpha_1 + a_2\cos\alpha_2 + ... + a_n\cos\alpha_n = 0$

and $a_1\sin\alpha_1 + a_2\sin\alpha_2 + ... + a_n\sin\alpha_n = 0$

$\Rightarrow\ f(x) = 0\ \forall\ x$

43. (b, d) : Given $2f(\sin x) + f(\cos x) = x$

Replace x by $\dfrac{\pi}{2} - x$,

$\Rightarrow\ 2f(\cos x) + f(\sin x) = \dfrac{\pi}{2} - x$

$\Rightarrow\ f(\sin x) = x - \dfrac{\pi}{6} \Rightarrow f(x) = \sin^{-1}x - \dfrac{\pi}{6}$

$\therefore$ Domain of $f(x)$ is $[-1, 1]$ and range of $f^{-1}(x)$ is $\left[\dfrac{-\pi}{2}-\dfrac{\pi}{6}, \dfrac{\pi}{2}-\dfrac{\pi}{6}\right]$ *i.e.*, $\left[\dfrac{-2\pi}{3}, \dfrac{\pi}{3}\right]$

44. (a, b, c, d) : $f\left(\frac{7}{2}\right) = -0.5$, $g\left(\frac{7}{2}\right) = 0.5$, $g(3) = 0$, $f(0) = 3$; $f(2) = -1$, $g(2) = -1$; $f(4) = 0$; $g(4) = 26$

45. (b) : $\because ff^{-1}(x) = x$

$2^{(f^{-1}(x))^4 - 4(f^{-1}(x))^2} = x$

$\Rightarrow (f^{-1}(x))^4 - 4(f^{-1}(x))^2 - \log_2 x = 0$

$\therefore (f^{-1}(x))^2 = 2 + \sqrt{4+\log_2 x}$

$\therefore$ Range of $f^{-1}(x)$ is $[2, \infty)$.

$\therefore f^{-1}(x) = \sqrt{2+\sqrt{4+\log_2 x}}$

$\therefore f^{-1}(x) > 0$

46. (a) : $g(x) = \dfrac{\sin x + 4}{\sin x - 2}$

$\Rightarrow g'(x) = \dfrac{-6\cos x}{(\sin x - 2)^2} \geq 0 \quad \because x \in \left[\frac{\pi}{2}, \pi\right]$

$\Rightarrow$ $g(x)$ is increasing function, hence one-one function.

$\therefore$ Range is $\left[g\left(\frac{\pi}{2}\right), g(\pi)\right]$ lie $[-5, -2]$.

47. (A)→(q), (B)→(q, s, t), (C)→(p, q, r, t), (D)→(q, r)

(A) $\dfrac{1-x^2}{x} + |x| = \left|\dfrac{1-x^x}{x} + x\right| = \left|\dfrac{1}{x}\right|$

$\Rightarrow \dfrac{1-x^2}{x} \cdot x \geq 0 \Rightarrow x^2 - 1 \leq 0,\ x \neq 0$

$\Rightarrow x \in [-1, 1] - \{0\}$

(B) Slope > 0 $\Rightarrow \dfrac{-(a+1)}{a} > 0 \Rightarrow a \in (-1, 0)$

(C) $a - a\cos^2 x + |\cos x| - 2a = 0$

$\Rightarrow a = \dfrac{|\cos x|}{1+\cos^2 x} = \dfrac{1}{|\cos x| + \dfrac{1}{|\cos x|}}$

$|\cos x| \in [0, 1] \Rightarrow |\cos x| + \dfrac{1}{|\cos x|} \in [2, \infty) \Rightarrow a \in \left(0, \dfrac{1}{2}\right]$

(D) If $x \neq n\pi$

$S = 1 + 2\cos^2 x + 3\cos^4 x + ...$ (1)

$\cos^2 xS = \cos^2 x + \cos^4 x + ...$ (2),

Subtracting eq. (1) and eq. (2), we get

$\sin^2 xS = 1 + \cos^2 x + \cos^4 x + ...$

$\sin^2 xS = \dfrac{1}{1-\cos^2 x} \Rightarrow S = \dfrac{1}{\sin^4 x} \Rightarrow f(x) = \sin^4 x$

For $x = n\pi$, $f(x)$ is not difined, Range of $f(x) = (0, 1]$

48. (3) : From the given equation

$f(x + y) = f(x) \cdot f(y)$, we have $f(x) = e^{kx}$ ...(i)

Where k is a constant.

Putting $x = 1$, $f(1) = e^k \Rightarrow 2 = e^k$. Hence from (i), $f(x) = 2^x$.

This can also be obtained by putting $y = 1$ so that $F(x + 1) = f(x) \cdot f(1) = 2f(x) \Rightarrow f(x) = 2f(x - 1)$.

Putting successively $x - 1, x - 2, x - 3, ..., 2$ for x in the above and multiplying them, we get $f(x) = 2^x$.

Now, $\sum_{r=1}^{n} f(c+r) = f(c + 1) + f(c + 2) + ... + f(c + n)$

$= 2^{c+1} + 2^{c+2} + ... + 2^{c+n}$

$= 2^c \cdot 2 + 2^c \cdot 2^2 + ... + 2^c \cdot 2^n$

$= 2^c \cdot 2\{1 + 2 + 2^2 + ...$ to n terms$\}$

$= 2^c \cdot 2 \cdot \dfrac{1(2^n - 1)}{2-1} = 2^c \cdot 2(2^n - 1)$

$\Rightarrow 16(2^n - 1) = 2^{c+1}(2^n - 1)$

$\Rightarrow 2^{c+1} = 16 = 2^4 \Rightarrow c + 1 = 4 \therefore c = 3$.

49. (3) : $f(1, 1) = f(0, f(1, 0)) = f(0, f(0, 1)) = f(0, 2) = 3$

50. (3) : $f(x) \times f'(-x) = f(-x) \times f'(x)$

$\Rightarrow f'(x) \times f(-x) - f(x) \times f'(-x) = 0$

$\Rightarrow \dfrac{d}{dx}[f(x)f(-x)] = 0$

$\Rightarrow f(x)\,f(-x) = k$

Given, $(f(0))^2 = k = 9 \quad \Rightarrow k = 9$

Then $\quad f(3)\,f(-3) = 9 \quad \Rightarrow f(-3) = 3$

51. (1) : $[x] = \dfrac{5 \pm \sqrt{25 + 4\sin x - 24}}{2 \cdot 1} = \dfrac{5 \pm \sqrt{1 + 4\sin x}}{2}$

$-1 \leq \sin x \leq 1$; $\quad -4 \leq 4\sin x \leq 4$; $\quad -3 \leq 1 + 4\sin \leq 5$

$0 \leq \sqrt{1 + 4\sin x} \leq 5$

$\Rightarrow [x]$ is an integer $\Leftrightarrow \sin x = 0$

$\Rightarrow [x] = 3 \quad \Rightarrow \quad x = \pi$

52. (5) : Let $f(x) = e^{mx}$ be a solution.

Then $\quad f(x + 2) = e^{m(x+2)} = e^{mx} \cdot e^{2m}$,

$f(x + 1) = e^{mx} \cdot e^m$

Therefore, $f(x + 2) - 5f(x + 1) + 6f(x) = 0$

$\Rightarrow e^{mx} \cdot e^{2m} - 5e^{mx} \cdot e^m + 6e^{mx} = 0$

$\Rightarrow e^{mx}[e^{2m} - 5e^m + 6] = 0$

$\Rightarrow e^{2m} - 5e^m + 6 = 0; \quad e^{mx} \neq 0$

Let $e^m = u$. Then from the above equation,

$u^2 - 5u + 6 = 0 \Rightarrow (u - 2)(u - 3) = 0$

$u = 2$ or $u = 3$.

$\Rightarrow e^m = 2$ or $e^m = 3$

$\Rightarrow e^{mx} = 2^x$ or $e^{mx} = 3^x$

Hence, $f(x) = A \cdot 2^x + B \cdot 3^x$.

53. (2) : Given, $f(1) + 2f(2) + 3f(3) + ... + nf(n)$

$= n(n + 1)f(n)$ ...(i)

Replacing n by $(n + 1)$, we get

$f(1) + 2f(2) + 3f(3) + ... + nf(n) + (n + 1)f(n + 1)$

$= (n + 1)(n + 2)f(n + 1)$ ...(ii)

Subtracting (i) from (ii), we get

$(n + 1)f(n + 1) = (n + 1)(n + 2)f(n + 1) - n(n + 1) f(n)$

$\Rightarrow \; f(n + 1) = (n + 2)f(n + 1) - nf(n)$

$\Rightarrow \; nf(n) = (n + 2)f(n + 1) - f(n + 1)$

$= (n + 2 - 1)f(n + 1) = (n + 1)f(n + 1).$

Thus, we have $2f(2) = 3f(3) = ... = nf(n)$.

Hence, from (1), $f(1) + (n - 1)nf(n) = n(n + 1)f(n)$

$\Rightarrow \; f(n)\{n(n + 1) - n(n - 1)\} = f(1) = 1$

$\Rightarrow \; (n^2 + n - n^2 + n)f(n) = 1$

$\Rightarrow \; 2nf(n) = 1$

$\Rightarrow \; f(n) = \dfrac{1}{2n} \quad \therefore f(200) = \dfrac{1}{2\times 200} = \dfrac{1}{400}$

54. (2) : Given $f(u + v) + f(u - v) = 2f(u) \cos v$ ...(i)

Putting $u = 0$ and $v = x$ in (i), we get

$f(x) + f(-x) = 2f(0) \cos x$ ...(ii)

$= 2a \cos x$ where $a = f(0)$

Now, putting $u = \dfrac{\pi}{2} - x$ and $v = \dfrac{\pi}{2}$ in (i), we get

$f(\pi - x) + f(-x) = 0 \because \cos\dfrac{\pi}{2} = 0.$

Also, on putting $u = \dfrac{\pi}{2}$ and $v = \dfrac{\pi}{2} - x$ in (i), we get

$f(\pi - x) + f(x) = 2f\left(\dfrac{\pi}{2}\right)\cos\left(\dfrac{\pi}{2} - x\right) = 2b \sin x$ ...(iii)

where $b = f\left(\dfrac{\pi}{2}\right)$

Now, adding (ii) and (iii), we get

$2f(x) - f(-x) + f(-x) = 2a \cos x + 2b \sin x$

$\Rightarrow \; 2f(x) = 2a \cos x + 2b \sin x$ (from (iii))

Hence $f(x) = a \cos x + b \sin x$.

55. (2) : Given, $f(x + y + 1) = \left(\sqrt{f(x)} + \sqrt{f(y)}\right)^2$

Putting $y = 0$, we get $f(x + 1) = \left(\sqrt{f(x)} + 1\right)^2$;

Putting $x = 0$, we have $f(1) = (1 + 1)^2 = 2^2$;

Putting $x = 1$, $f(2) = (2 + 1)^2 = 3^2$;

Putting $x = 2$, $f(3) = \left(\sqrt{f(2)} + 1\right)^2 = (3 + 1)^2 = 4^2$ and so on.

Proceeding in this way, we get $f(x) = (x + 1)^2$.

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CLASS XII Series 5

ACE YOUR WAY CBSE

Integrals | Application of Integrals

IMPORTANT FORMULAE

INTEGRALS

- $\int dx = x + C$, *where 'C' is an arbitrary constant*
- $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$ *where* $n \neq -1$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \dfrac{a^x}{\log_e a} + C$, *where* $a > 0$
- $\int \dfrac{1}{x} dx = \log_e |x| + C$, *where* $x \neq 0$
- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$
- $\int \sec x \tan x \, dx = \sec x + C$
- $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$
- $\int \tan x \, dx = \log|\sec x| + C$ or $-\log|\cos x| + C$
- $\int \cot x \, dx = \log|\sin x| + C$ or $-\log \operatorname{cosec} x + C$
- $\int \sec x \, dx = \log|\sec x + \tan x| + C$
  $= \log\left|\tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)\right| + C$
- $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C$
  $= \log\left|\tan\dfrac{x}{2}\right| + C$
- $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$ or $-\cos^{-1} x + C$, *where* $|x| < 1$
- $\int \dfrac{dx}{1+x^2} = \tan^{-1} x + C$ or $-\cot^{-1} x + C$
- $\int \dfrac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$ or $-\operatorname{cosec}^{-1} x + C$
- $\int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$
- $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right| + C$
- $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right| + C$
- $\int \dfrac{dx}{\sqrt{x^2+a^2}} = \log\left|x + \sqrt{x^2+a^2}\right| + C$
- $\int \dfrac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right| + C$
- $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C, |x| < a$
- $\int \sqrt{a^2+x^2}\, dx = \dfrac{x}{2}\sqrt{a^2+x^2} + \dfrac{a^2}{2}\log|x + \sqrt{a^2+x^2}| + C$
- $\int \sqrt{x^2-a^2}\, dx = \dfrac{x}{2}\sqrt{x^2-a^2} - \dfrac{a^2}{2}\log|x + \sqrt{x^2-a^2}| + C$
- $\int \sqrt{a^2-x^2}\, dx = \dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right) + C$

- *By parts (ILATE)*

$$\int I \cdot II\, dx = I \cdot \int II\, dx - \int \left(\frac{d}{dx}(I) \cdot \int II\, dx \right) dx$$

(Here I and II functions are choosen on the basis of ILATE)

INTEGRATION BY PARTIAL FRACTIONS

Rational form	Partial form
$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{(x-a)}+\frac{B}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)}+\frac{B}{(x-b)}+\frac{C}{(x-c)}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)^3(x-b)}$	$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}+\frac{D}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)}+\frac{Bx+C}{x^2+bx+c}$ *where* x^2+bx+c *cannot be factorised further*

Properties of Definite Integrals

- $\int_a^b f(x)dx = \int_a^b f(t)dt$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$. *In particular,* $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
- $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$
- $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(2a-x)=f(x) \\ 0, \text{ if } f(2a-x)=-f(x) \end{cases}$
- $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x), \text{ if } f(-x)=f(x) \text{ [even function]} \\ 0, \text{ if } f(-x)=-f(x) \text{ [odd function]} \end{cases}$
- $\int e^x[f(x)+f'(x)]dx = e^x f(x) + C$
- $\int (f(x))^n f'(x)dx = \frac{(f(x))^{n+1}}{n+1} + C$
- $\int \frac{f'(x)}{f(x)}dx = \log(f(x)) + C$

APPLICATION OF INTEGRALS

- *The area of a region bounded by* $y^2 = 4ax$ *and* $x^2 = 4by$ *is* $\frac{16ab}{3}$ *sq. units.*
- *The area of a region bounded by* $y^2 = 4ax$ *and* $y = mx$ *is* $\frac{8a^2}{3m^3}$ *sq. units.*
- *The area of a region bounded by* $y^2 = 4ax$ *and its latus rectum is* $\frac{8a^2}{3}$ *sq. units.*
- *The area of a region bounded by one arc of sinax or cosax and x-axis is* $\frac{2}{a}$ *sq. units.*
- *Area of an ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ *is* πab *sq. units.*
- *The area of a region bounded by* $y = ax^2 + bx + c$ *and x-axis is* $\frac{(b^2-4ac)^{\frac{3}{2}}}{6a^2}$ *sq. units.*

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Evaluate $\int e^{3\log x}(x^4)\,dx$

2. Find $\int_{-1}^{1}|x|\,dx$

3. Evaluate $\int\limits_{-\pi/2}^{\pi/2}\log\left(\dfrac{px^2+qx+r}{px^2-qx+r}\right)dx$

4. Solve $\int\dfrac{3}{\sqrt{3x-2}}\,dx$

5. Evaluate $\int(2^x+2^{-x})^2\,dx$

SHORT ANSWER TYPE

6. Find the area bounded by the curves $y = x$ and $y = x^3$.

7. Evaluate $\int\dfrac{(a^x+b^x)^2}{a^x b^x}dx$

8. Find the value of $\int_{-2008}^{2008}\dfrac{f'(x)+f'(-x)}{(2008)^x+1}dx$

9. Evaluate $\int\dfrac{dx}{\sqrt{1-2x-x^2}}$

10. Find the area of the region bounded by the curve $x^2 = y$, and the line $y = 4$.

LONG ANSWER TYPE - I

11. Find $\int\dfrac{2x-3}{(x^2-1)(2x+3)}\,dx$

12. Find the value of $\int\limits_{e^{-1}}^{e^2}\left|\dfrac{\log x}{x}\right|dx$

13. Evaluate $\int\dfrac{2\sin x+3\cos x}{3\sin x+4\cos x}dx$

14. Evaluate $\int_{-1}^{1}e^x dx$ as the limit of a sum.

15. Using integration, find the area of the region bounded by the triangle whose vertices are (–1, 2), (1, 5) and (3, 4).

LONG ANSWER TYPE - II

16. Evaluate $\int\dfrac{\sqrt{x^2+1}(\log(x^2+1)-2\log x)}{x^4}dx,\ x>0$

17. Show that

(i) $\int\limits_0^{\pi/2} f(\sin 2x)\sin x\,dx = \int\limits_0^{\pi/2} f(\sin 2x)\cos x\,dx$

(ii) $\int\limits_0^{\pi/2} f(\sin 2x)\sin x\,dx = \sqrt{2}\int\limits_0^{\pi/4} f(\cos 2x)\cos x\,dx$

18. Find the smaller of the two areas in which the circles $x^2 + y^2 = 4$ is divided by the parabola $y^2 = 3(2x - 1)$.

19. Evaluate the following definite integrals :

(i) $\int\limits_0^{a}\dfrac{dx}{x+\sqrt{a^2-x^2}}$

(ii) $\int\limits_0^{1}\cot^{-1}(1-x+x^2)dx$

20. Find the area enclosed between the curves $y = \sin x$ and $y = \cos x$ that lies between the lines $x = 0$ and $x = \dfrac{\pi}{2}$.

SOLUTIONS

1. $\int e^{3\log x}(x^4)dx = \int x^3\cdot x^4 dx \quad (\because\ e^{\log x} = x)$

$= \int x^7 dx = \dfrac{x^8}{8}+C$

2. We have, $\int_{-1}^{1}|x|\,dx = \int_{-1}^{0}-x\,dx+\int_0^1 x\,dx$

$$= -\left[\frac{x^2}{2}\right]_{-1}^{0}+\left[\frac{x^2}{2}\right]_0^1 = -\left(0-\frac{1}{2}\right)+\left(\frac{1}{2}-0\right)=1$$

3. Let $f(x)=\log\left(\dfrac{px^2+qx+r}{px^2-qx+r}\right)$

$$\Rightarrow\quad f(-x)=\log\left(\frac{p(-x)^2+q(-x)+r}{p(-x)^2-q(-x)+r}\right)$$

$$=\log\left(\frac{px^2-qx+r}{px^2+qx+r}\right)$$

$$=\log\left(\frac{px^2+qx+r}{px^2-qx+r}\right)^{-1} = -\log\left(\frac{px^2+qx+r}{px^2-qx+r}\right) = -f(x)$$

$\Rightarrow$ $f(x)$ is an odd function.

$$\therefore\quad \int\limits_{-\pi/2}^{\pi/2}\log\left(\frac{px^2+qx+r}{px^2-qx+r}\right)dx=0$$

4. We have, $\int\dfrac{3}{\sqrt{3x-2}}dx = 3\int(3x-2)^{-1/2}\,dx$

$$= 3\cdot\frac{(3x-2)^{1/2}}{\frac{1}{2}\cdot 3}+C = 2\sqrt{3x-2}+C.$$

5. We have, $\int (2^x + 2^{-x})^2\, dx = \int (2^{2x} + 2^{-2x} + 2)\, dx$

$$= \frac{2^{2x}}{(\log 2)\times 2} + \frac{2^{-2x}}{(\log 2)\,(-\,2)} + 2\cdot x + C$$

$$= \frac{1}{2\log 2}(2^{2x} - 2^{-2x}) + 2x + C$$

6. The given curves are $y = x$ ...(i) and $y = x^3$ ...(ii)

Solving (i) and (ii), we get

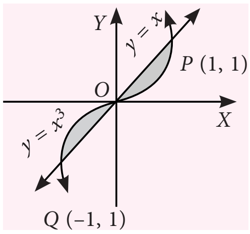
$x^3 = x \Rightarrow x(x^2 - 1) = 0$

$\Rightarrow x = 0, 1, -1$

When $x = 0$, $y = 0$;

when $x = 1$, $y = 1$ and

when $x = -1$, $y = -1$.



So, $O(0, 0)$, $P(1, 1)$ and $Q(-1, -1)$ are points of intersection of (i) and (ii).

$$\therefore \text{ Required area } = 2\int_0^1 (x - x^3)dx = 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= 2\left[\left(\frac{1}{2} - \frac{1}{4}\right) - (0 - 0)\right] = 2\times\frac{1}{4} = \frac{1}{2} \text{ sq. units}$$

7. We have, $\int \frac{(a^x + b^x)^2}{a^x b^x}dx = \int \frac{a^{2x} + b^{2x} + 2a^x b^x}{a^x b^x}dx$

$$= \int\left(\left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x + 2\right)dx = \frac{\left(\frac{a}{b}\right)^x}{\log\frac{a}{b}} + \frac{\left(\frac{b}{a}\right)^x}{\log\frac{b}{a}} + 2x + C, a \neq b$$

8. $I = \int_{-2008}^{2008} \frac{f'(-x) + f'(x)}{(2008)^{-x} + 1}dx$

$$= \int_{-2008}^{2008} \frac{f'(-x) + f'(x)}{1 + (2008)^x} \times (2008)^x\, dx$$

$$\therefore\ 2I = \int_{-2008}^{2008} f'(-x) + f'(x)dx$$

$$= 2\int_0^{2008} f'(x) + f'(-x)dx$$

$$I = [f(x) - f(-x)]_0^{2008} = f(2008) - f(-2008)$$

9. $I = \int \frac{dx}{\sqrt{1 - (x^2 + 2x)}} = \int \frac{dx}{\sqrt{2 - (x^2 + 2x + 1)}}$

$$= \int \frac{dx}{\sqrt{2 - (1 + x)^2}} = \int \frac{dx}{\sqrt{(\sqrt{2})^2 - (1 + x)^2}} \quad ...(i)$$

Let $z = 1 + x$, then $dz = dx$

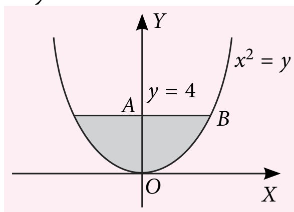
From (i),

$$I = \int \frac{dz}{\sqrt{(\sqrt{2})^2 - z^2}} = \sin^{-1}\frac{z}{\sqrt{2}} + c = \sin^{-1}\left(\frac{1 + x}{\sqrt{2}}\right) + c$$

10. Given curves are $x^2 = y$ and $y = 4$

$\therefore$ Required area

= 2 area $OABO$



$$= 2\int_0^4 x dy$$

$$= 2\int_0^4 \sqrt{y} dy$$

$$= 2\cdot\frac{2}{3}\left[y^{3/2}\right]_0^4 = \frac{4}{3}(8 - 0) = \frac{32}{3} \text{ sq. units}$$

11. Let $\frac{2x - 3}{(x - 1)\,(x + 1)\,(2x + 3)}$

$$= \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{2x + 3}$$

$\Rightarrow 2x - 3 = A\,(x + 1)\,(2x + 3) + B(x - 1)\,(2x + 3)$
$+ C\,(x - 1)\,(x + 1)$

$\Rightarrow 2x - 3 = A\,(2x^2 + 5x + 3) + B\,(2x^2 + x - 3) + C\,(x^2 - 1)$

Equating the coefficients of x^2, x and constant terms, we get

$2A + 2B + C = 0$, $5A + B = 2$ and $3A - 3B - C = -3$

Solving these equations, we get

$$A = -\frac{1}{10},\ B = \frac{5}{2} \text{ and } C = -\frac{24}{5}$$

$$\therefore\ \frac{2x - 3}{(x^2 - 1)(2x + 3)} = \frac{-\frac{1}{10}}{x - 1} + \frac{\frac{5}{2}}{x + 1} + \frac{-\frac{24}{5}}{2x + 3}$$

$$\int \frac{2x - 3}{(x^2 - 1)(2x + 3)}dx = -\frac{1}{10}\int\frac{dx}{x - 1}$$

$$+ \frac{5}{2}\int\frac{dx}{x + 1} - \frac{24}{5}\int\frac{dx}{2x + 3}$$

$$= -\frac{1}{10}\log|x - 1| + \frac{5}{2}\log|x + 1| - \frac{24}{5}\frac{\log|2x + 3|}{2} + C$$

$$= -\frac{1}{10}\log|x - 1| + \frac{5}{2}\log|x + 1| - \frac{12}{5}\log|2x + 3| + C$$

12. The value of $\int_{e^{-1}}^{e^2}\left|\frac{\log x}{x}\right|dx$

$\log x = 0$ when $x = 1$ and hence negative in $\left[\frac{1}{e}, 1\right)$ and positive in $(1, e^2]$

MPP-5 CLASS XII ANSWER KEY

1. (a)	2. (b)	3. (c)	4. (c)	5. (b)
6. (b)	7. (b,c,d)	8. (a,b,c)	9. (a,d)	10. (a)
11. (a,c,d)	12. (a,b,d,)	13. (a,c)	14. (b)	15. (c)
16. (b)	17. (3)	18. (1)	19. (3)	20. (1)

$$\therefore \quad I=\int_{1/e}^{1}\left|\frac{\log x}{x}\right|dx+\int_{1}^{e^2}\left|\frac{\log x}{x}\right|dx$$

$$=\int_{1/e}^{1}\frac{-\log x}{x}dx+\int_{1}^{e^2}\frac{\log x}{x}dx$$

$$=\left[-\frac{1}{2}(\log x)^2\right]_{\frac{1}{e}}^{1}+\left[\frac{1}{2}(\log x)^2\right]_{1}^{e^2}=\frac{1}{2}(-1)^2+\frac{4}{2}-0=\frac{5}{2}$$

13. Let 2 sin x + 3 cos x = l (3 sin x + 4 cos x) + m (3 cos x – 4 sin x).

Equating the coefficients of sin$x$ and cos$x$ on both sides, we get

$2 = 3l - 4m$ and $3 = 4l + 3m$.

On solving, we get $l=\frac{18}{25}$ and $m=\frac{1}{25}$

$$\therefore \quad \int\frac{2\sin x+3\cos x}{3\sin x+4\cos x}dx$$

$$=\int\frac{l(3\sin x+4\cos x)+m(3\cos x-4\sin x)}{3\sin x+4\cos x}dx$$

$$=l\int 1\,dx+m\int\frac{3\cos x-4\sin x}{3\sin x+4\cos x}dx$$

(put $3\sin x+4\cos x=t \Rightarrow (3\cos x-4\sin x)\,dx=dt$)

$$=lx+m\int\frac{dt}{t}=lx+m\log|t|+C$$

$$=\frac{18}{25}x+\frac{1}{25}\log|3\sin x+4\cos x|+C$$

14. $\int_a^b f(x)\,dx=\lim_{h\to 0}\sum_{r=1}^{n}h\,f(a+rh)$ ...(i)

Where, $nh = b - a$, and $n\to\infty$

Here, $f(x) = e^x$, $a = -1$, $b = 1$

$\therefore$ $f(a + rh) = e^{a+rh} = e^a\cdot e^{rh}$

Also, $nh = b - a = 1 - (-1) = 2$

From (i), $\int_{-1}^{1}e^x dx=\lim_{h\to 0}\sum_{r=1}^{n}h\,e^a\cdot e^{rh}$

$$=\lim_{h\to 0}\sum_{r=1}^{n}h\,e^{-1}\cdot e^{rh}=\lim_{h\to 0}e^{-1}\cdot h\sum_{r=1}^{n}e^{rh}$$

$$=\lim_{h\to 0}\frac{1}{e}\cdot h(e^h+e^{2h}+e^{3h}+\ldots+e^{nh})$$

$$=\lim_{h\to 0}\frac{h}{e}\left[\frac{e^h\{1-(e^h)^n\}}{1-e^h}\right]=\lim_{h\to 0}\frac{e^h}{e}\left[\frac{h(1-e^{nh})}{1-e^h}\right]$$

$$=\lim_{h\to 0}\frac{e^h}{e}\left[h\cdot\frac{e^{nh}-1}{e^h-1}\right]=\lim_{h\to 0}\frac{e^h}{e}\left[\frac{e^2-1}{\frac{e^h-1}{h}}\right]$$

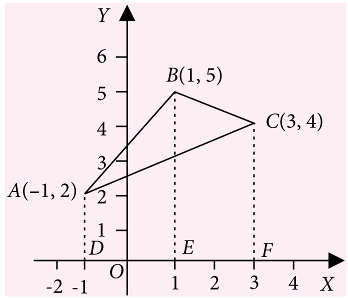
$$=\frac{1}{e}\cdot e^0\cdot\frac{e^2-1}{1}\left[\because \lim_{h\to 0}\frac{e^h-1}{h}=1\right]=e-\frac{1}{e}$$

15. Let the vertices of the given triangle be A (–1, 2), B (1, 5) and C (3, 4).

The equations of the sides AB, BC and CA respectively are:

$y=\frac{3x+7}{2}$ ...(i); $\quad y=\frac{-x+11}{2}$ ...(ii)

and $y=\frac{x+5}{2}$ ...(iii)



$\therefore$ Required area = Area of region bounded by trap. $ADEB$ + Area of region bounded by trap. $BEFC$ – Area of region bounded by trap. $ADFC$

$$=\int_{-1}^{1}\frac{3x+7}{2}dx+\int_{1}^{3}\frac{-x+11}{2}dx-\int_{-1}^{3}\frac{x+5}{2}dx$$

$$=\frac{1}{2}\left[\frac{3x^2}{2}+7x\right]_{-1}^{1}+\frac{1}{2}\left[-\frac{x^2}{2}+11x\right]_{1}^{3}-\frac{1}{2}\left[\frac{x^2}{2}+5x\right]_{-1}^{3}$$

$$=\frac{1}{2}\left[\left(\frac{3}{2}+7\right)-\left(\frac{3}{2}-7\right)\right]+\frac{1}{2}\left[\left(-\frac{9}{2}+33\right)-\left(-\frac{1}{2}+11\right)\right]$$

$$-\frac{1}{2}\left[\left(\frac{9}{2}+15\right)-\left(\frac{1}{2}-5\right)\right]$$

$$=\frac{1}{2}(14+22-4)-\frac{1}{2}(4+20)$$

$$=\frac{1}{2}(32)-\frac{1}{2}(24)=16-12=4\text{ sq. units.}$$

16. $\int\frac{\sqrt{x^2+1}(\log(x^2+1)-2\log x)}{x^4}dx$

$$=\int\frac{x\sqrt{1+\frac{1}{x^2}}(\log(x^2+1)-\log x^2)}{x^4}dx$$

$$= \int \left(1+\frac{1}{x^2}\right)^{1/2} \log\left(1+\frac{1}{x^2}\right)\cdot\frac{1}{x^3}dx$$

$$\left(\text{Put } 1+\frac{1}{x^2}=t \Rightarrow -\frac{2}{x^3}dx = dt \;\Rightarrow \frac{1}{x^3}dx - \frac{1}{2}dt\right)$$

$$= \int t^{1/2}\log t\left(-\frac{1}{2}\right)dt = -\frac{1}{2}\int \log t\cdot t^{1/2}dt$$

$$= -\frac{1}{2}\left[\log t\cdot\frac{t^{3/2}}{3/2} - \int \frac{1}{t}\cdot\frac{t^{3/2}}{3/2}dt\right] \qquad \text{(Integrate by parts)}$$

$$= -\frac{1}{3}t^{3/2}\log t + \frac{1}{3}\int t^{1/2}dt = -\frac{1}{3}t^{3/2}\log t + \frac{1}{3}\cdot\frac{t^{3/2}}{3/2}+C$$

$$= \frac{1}{9}t^{3/2}(2-3\log t)+C$$

$$= \frac{1}{9}\left(1+\frac{1}{x^2}\right)^{3/2}\left[2-3\log\left(1+\frac{1}{x^2}\right)\right]+C$$

17. (i) $\int_0^{\pi/2} f(\sin 2x)\sin x\, dx$

$$= \int_0^{\pi/2} f\left(\sin 2\left(\frac{\pi}{2}-x\right)\right)\sin\left(\frac{\pi}{2}-x\right)dx$$

$$= \int_0^{\pi/2} f(\sin(\pi-2x))\cos x\,dx = \int_0^{\pi/2} f(\sin 2x)\cos x\,dx$$

(ii) Put $x = \frac{\pi}{4}-t \Rightarrow t = \frac{\pi}{4}-x \Rightarrow dx = -dt$

When $x = 0$,

$t = \frac{\pi}{4}-0 = \frac{\pi}{4}$ and when $x = \frac{\pi}{2}$, $t = \frac{\pi}{4}-\frac{\pi}{2} = -\frac{\pi}{4}$

$$\therefore \int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \int_{\pi/4}^{-\pi/4} f\left(\sin 2\left(\frac{\pi}{4}-t\right)\right)\sin\left(\frac{\pi}{4}-t\right)(-dt)$$

$$= -\int_{\pi/4}^{-\pi/4} f(\cos 2t)\left(\sin\frac{\pi}{4}\cos t - \cos\frac{\pi}{4}\sin t\right)dt$$

$$= \int_{-\pi/4}^{\pi/4} f(\cos 2t)\left(\frac{1}{\sqrt{2}}\cos t - \frac{1}{\sqrt{2}}\sin t\right)dt$$

$$= \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f(\cos 2t)\cos t\,dt - \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f(\cos 2t)\sin t\,dt$$

$$= \frac{1}{\sqrt{2}}\cdot 2\int_0^{\pi/4} f(\cos 2t)\cos t\,dt - \frac{1}{\sqrt{2}}\cdot 0$$

($\because f(\cos 2t)\cos t$ is an even function and $f(\cos 2t)\sin t$ is an odd function)

$$= \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$$

18. The given circle is $x^2 + y^2 = 4$ ...(i)

Its centre is (0, 0) and radius = 2.

The given parabola is $y^2 = 3(2x - 1)$ ...(ii)

$$\Rightarrow \quad y^2 = 6\left(x-\frac{1}{2}\right) \qquad \text{...(iii)}$$

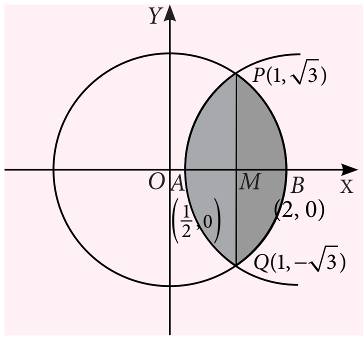
Solving (i) and (ii), we get

$x^2 + 3(2x - 1) = 4 \Rightarrow x^2 + 6x - 7 = 0$

$\Rightarrow (x + 7)(x - 1) = 0 \Rightarrow x = -7, 1$

but $x \ge \frac{1}{2}$, from (iii) we get, $x = 1$

When $x = 1$, $y^2 = 3 \Rightarrow y = \pm\sqrt{3}$



$\therefore$ $P(1, \sqrt{3})$, and $Q(1, -\sqrt{3})$ are points of intersection of (i) and (ii)

So, required area = 2 · area of the region PAB

$$= 2\int_{1/2}^{1}\sqrt{3(2x-1)}\,dx + 2\int_1^2\sqrt{4-x^2}\,dx$$

$$= 2\cdot\sqrt{3}\left[\frac{(2x-1)^{3/2}}{\frac{3}{2}\cdot 2}\right]_{1/2}^{1} + 2\cdot\left[\frac{x\sqrt{4-x^2}}{2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \frac{2\sqrt{3}}{3}(1-0)+(0+4\sin^{-1}(1))-\left(\sqrt{3}+4\sin^{-1}\left(\frac{1}{2}\right)\right)$$

$$= \frac{2\sqrt{3}}{3}+4\cdot\frac{\pi}{2}-\left(\sqrt{3}+4\cdot\frac{\pi}{6}\right)$$

$$= \sqrt{3}\left(\frac{2}{3}-1\right)+2\pi-\frac{2}{3}\pi = -\frac{\sqrt{3}}{3}+\frac{4\pi}{3}$$

$$= \frac{1}{3}(4\pi-\sqrt{3})\text{ sq. units}$$

19. **(i)** Let $I = \int_0^1 \frac{dx}{x + \sqrt{a^2 - x^2}}$

Put $x = a\sin t \Rightarrow dx = a\cos t\, dt$

When $x = 0$, $t = 0$ and when $x = a$, $t = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \int_0^{\pi/2} \frac{a\cos t}{a\sin t + a\cos t} dt$$

$$= \int_0^{\pi/2} \frac{\cos t}{\sin t + \cos t} dt \qquad \text{...(i)}$$

Then by using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get

$$I = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-t\right)}{\sin\left(\frac{\pi}{2}-t\right)+\cos\left(\frac{\pi}{2}-t\right)} dt = \int_0^{\pi/2} \frac{\sin t}{\cos t + \sin t} dt \qquad \text{...(ii)}$$

On adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\cos t + \sin t}{\sin t + \cos t} dt = \int_0^{\pi/2} 1 dt = [t]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

(ii) We have, $\cot^{-1}(1 - x + x^2) = \tan^{-1}\left(\frac{1}{1-x+x^2}\right)$

$$= \tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right) = \tan^{-1} x + \tan^{-1}(1-x)$$

$$\Rightarrow \int_0^1 \cot^{-1}(1-x+x^2)dx = \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-x) dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-(1-x))\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1} x\, dx$$

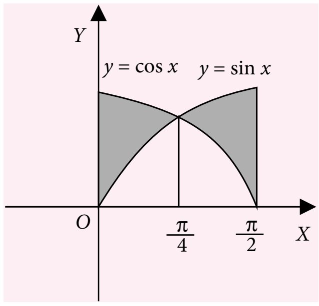
$$= 2\int_0^1 \tan^{-1} x \cdot 1\, dx \qquad \text{(integrate by parts)}$$

$$= 2\left(\left[(\tan^{-1} x \cdot x)\right]_0^1 - \int_0^1 \frac{1}{1+x^2} \cdot x\, dx\right)$$

$$= 2(1 \cdot \tan^{-1}(1) - 0) - \int_0^1 \frac{2x}{1+x^2} dx = 2 \cdot \frac{\pi}{4} - |\log(1+x^2)|_0^1$$

$$= \frac{\pi}{2} - (\log 2 - \log 1) = \frac{\pi}{2} - (\log 2 - 0) = \frac{\pi}{2} - \log 2$$

20. Given, curves are $y = \sin x$ and $y = \cos x$;



The curves cross each other at $\sin x = \cos x$ *i.e.*, at $\tan x = 1$ *i.e.*, at $x = \frac{\pi}{4}$.

$$\therefore \text{ Required area} = \int_0^{\pi/2} |\cos x - \sin x| dx$$

$$\int_0^{\pi/4} |\cos x - \sin x| dx + \int_{\pi/4}^{\pi/2} |\cos x - \sin x| dx$$

$$\int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

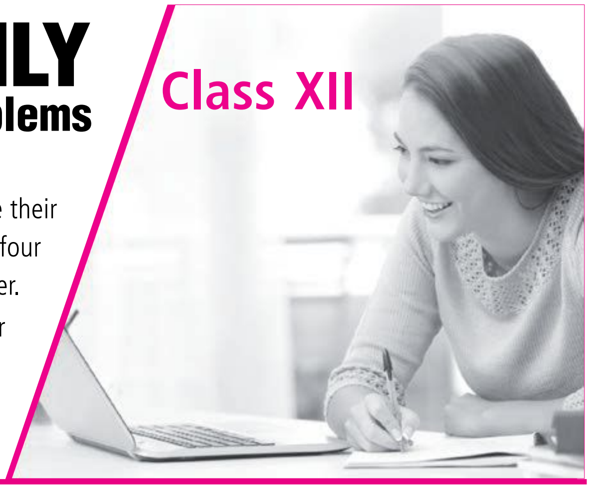
$$= \left[\sin x + \cos x\right]_0^{\pi/4} + \left[-\cos x - \sin x\right]_{\pi/4}^{\pi/2}$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)\right] - \left[(0+1) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{2}{\sqrt{2}} - 1 - 1 + \frac{2}{\sqrt{2}} = 2\sqrt{2} - 2 = 2\left(\sqrt{2} - 1\right) \text{ sq. units.}$$

MPP-5 MONTHLY Practice Problems

Class XII



This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Application of Derivatives

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. The coordinates of the point $P(x, y)$ lying in the first quadrant on the ellipse $x^2/8 + y^2/18 = 1$ so that the area of the triangle formed by the tangent at P and the coordinate axes is the smallest, are given by
(a) $(2, 3)$ (b) $(\sqrt{8}, 0)$
(c) $(\sqrt{18}, 0)$ (d) none of these

2. If $f(x) = x^{3/2}(3x - 10), x > 0$, then $f(x)$ is increasing in
(a) $(-\infty, -1) \cup (1, \infty)$ (b) $[2, \infty)$
(c) $(-\infty, -1) \cup [1, \infty)$ (d) $(-\infty, 0] \cup (2, \infty)$

3. The set of all values of a for which the function $f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right) x^5 - 3x + \log 5$ decreases for all real x is
(a) $\left(-3, \frac{5-\sqrt{27}}{2}\right) \cup (2, \infty)$
(b) $\left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$
(c) $(-\infty, \infty)$
(d) $[1, \infty)$.

4. If the function $f(x) = \cos|x| - 2ax + b$ increases along the entire number scale, the range of values of a is given by
(a) $a \le b$ (b) $a = b/2$
(c) $a \le -1/2$ (d) $a \ge 3/2$

5. The image of the interval $[-1, 3]$ under the maping $f(x) = 4x^3 - 12x$ is
(a) $[-2, 0]$ (b) $[-8, 72]$
(c) $[-8, 0]$ (d) none of these

6. By LMVT, which of the following is true for $x > 1$?
(a) $1 + x \ln x < x < 1 + \ln x$
(b) $1 + \ln x < x < 1 + x \ln x$
(c) $x < 1 + x \ln x < 1 + \ln x$
(d) $1 + \ln x < 1 + x \ln x < x$

One or More Than One Option(s) Correct Type

7. For the function $f(x) = x \cos \frac{1}{x}, x \ge 1$, which of the following is (are) correct?
(a) for at least one x in the interval $[1, \infty)$, $f(x + 2) - f(x) < 2$
(b) $\lim\limits_{x \to \infty} f'(x) = 1$
(c) for all x in the interval $[1, \infty]$, $f(x + 2) - f(x) > 2$
(d) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

8. Let $F : R \to R$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in R$. Then which of the following statements is (are) correct?
(a) $f'(1) < 0$ (b) $f(2) < 0$
(c) $f'(x) \ne 0$ for all $x \in (1, 3)$
(d) $f'(x) = 0$ for some $x \in (1, 3)$

9. If $\phi(x) = f(x) + f(2a - x)$ and $f''(x) > 0$, $a > 0$, $0 \le x \le 2a$, then
(a) $\phi(x)$ increases in $(a, 2a)$
(b) $\phi(x)$ increases in $(0, a)$
(c) $\phi(x)$ decreases in $(a, 2a)$
(d) $\phi(x)$ decreases in $(0, a)$

10. Let $f(x) = \begin{cases} x^3 + x^2 - 10x & -1 \le x < 0 \\ \sin x & 0 \le x < \pi/2 \\ 1 + \cos x & \pi/2 \le x \le \pi \end{cases}$
then $f(x)$ has

(a) local maxima at $x = \pi/2$
(b) local minima at $x = \pi/2$
(c) absolute minima at $x = 0$
(d) absolute maxima at $x = \pi/2$

11. Consider the function $f : R \to R$ given by

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}, \; 0 < a < 2.$$

If $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2}\,dt$, which of the following is (are) not true?
(a) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$
(b) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$
(c) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$
(d) $g'(x)$ does not change sign on $(-\infty, \infty)$

12. For $x > 1$, $y = \log_e x$ satisfies the inequality
(a) $x - 1 > y$ (b) $x^2 - 1 > y$
(c) $y > x - 1$ (d) $\frac{x-1}{x} < y$

13. Suppose $f'(x)$ exists for each x and $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then
(a) h is increasing whenever f is increasing
(b) h is increasing whenever f is decreasing
(c) h is decreasing whenever f is decreasing
(d) nothing can be said in general.

Comprehension Type

A window of fixed perimeter (including the base of the arch) is in the form of a rectangle surmounted by a semi-circle. The semi-circular portion is fitted with coloured glass, while the rectangular portion is fitted with clear glass. The clear glass transmits three times as much light per square metre as the coloured glass. Suppose that y is the length and x is the breadth of the rectangular portion and p the perimeter.

14. The ratio of the sides $y : x$ of the rectangle so that the window transmit the maximum light is
(a) $3 : 2$ (b) $6 : 6 + \pi$
(c) $6 + \pi : 6$ (d) $1 : 2$

15. If L is the total light transmitted, then critical points of L are
(a) $y = \frac{p}{6 + 3\pi/2}$ (b) $y = \frac{3p}{12 + 3\pi/2}$
(c) $y = \frac{3p}{12 + 5\pi/2}$ (d) $y = \frac{2p}{11 + 5\pi/2}$

Matrix Match Type

16. Match the following.

	Column I		Column II
P.	The function $f(x) = 2x^3 - 9x^2 - 24x + 7$ decreases on	1.	$(3, \infty)$
Q.	$f(x) = \frac{1+3x}{\sqrt{4+5x^2}}$ increases on	2.	$(-1, 4)$
R.	$f(x) = (x^2 - 2x) \log x - \frac{3}{2}x^2 + 4x$ increases on	3.	$(-\infty, 2]$
		4.	(e, ∞)

	P	Q	R
(a)	2	3	1
(b)	2	3	4
(c)	1	2	3
(d)	4	3	1

Integer Answer Type

17. If the point on $y = x \tan \alpha - \frac{ax^2}{2u^2 \cos^2 \alpha}$ $(\alpha > 0)$, where the tangent is parallel to $y = x$ has an ordinate $u^2/4a$, then $4 \sin^2 \alpha$ is equal to

18. If the greatest and least values of the functions $f(x) = \text{arc} \tan x - \frac{1}{2}\ln x$ on $\left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$ are G and L respectively, then $[G + L]$ = (where $[\cdot]$ is greater integer)

19. If $f(x) = |x - 1| + |x - 3| + |5 - x|$, $\forall\, x \in R$ is symmetrical about the line $x = \lambda$, then λ =

20. If $A > 0$, $B > 0$ and $A + B = \pi/3$ then the maximum value of $3 \tan A \tan B$ is

◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……
No. of questions correct ……
Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 177

JEE MAIN

1. If $x^2 - y^2 - 84y = 2012$, $x, y \in N$, then $x - 3y =$
(a) 2 (b) 3 (c) 6 (d) 7

2. Let $g(x) = \dfrac{(x-1)^n}{\log \cos^m (x-1)}; 0 < x < 2$, m and n are integers, $m \neq 0$, $n > 0$, and let p be the left hand derivative of $|x - 1|$ at $x = 1$. If $\lim_{x \to 1^+} g(x) = p$, then
(a) $n = 1, m = 1$ (b) $n = 1, m = -1$
(c) $n = 2, m = 2$ (d) $n > 2, m = n$

3. Given, $x = a\cos t\sqrt{\cos 2t}$ and $y = a\sin t\sqrt{\cos 2t}\,(a > 0)$, then $\left|\dfrac{(1+(dy/dx)^2)^{3/2}}{d^2y/dx^2}\right|$ at $\dfrac{\pi}{6}$ is given by
(a) $\dfrac{a}{3}$ (b) $a\sqrt{2}$ (c) $\dfrac{\sqrt{2}}{3a}$ (d) $\dfrac{\sqrt{2}a}{3}$

4. $\displaystyle\int \frac{\sin^3 x/2}{\cos\frac{x}{2}\sqrt{\cos^3 x + \cos^2 x + \cos x}}\,dx$ is equal to
(a) $\cos^{-1}\sqrt{\sec x + \tan x + 1} + c$
(b) $\tan^{-1}\sqrt{\sin x + \cos x + 1} + c$
(c) $\sin^{-1}\sqrt{\tan x + \sec x + 1} + c$
(d) $\tan^{-1}\sqrt{\cos x + \sec x + 1} + c$

5. The limiting points of the coaxial system of circles given by $x^2 + y^2 + 2gx + c + \lambda(x^2 + y^2 + 2fy + k) = 0$ subtend a right angle at the origin, if
(a) $-\dfrac{c}{g^2} - \dfrac{k}{f^2} = 2$ (b) $\dfrac{c}{g^2} + \dfrac{k}{f^2} = -2$
(c) $\dfrac{c}{g^2} - \dfrac{k}{f^2} = 2$ (d) $\dfrac{c}{g^2} + \dfrac{k}{f^2} = 2$

JEE ADVANCED

6. Five balls are to be placed in three boxes. Each can hold all the five balls. In how many different ways can we place the balls so that no box remains empty, if balls are different but boxes are identical?
(a) 25 (b) 15 (c) 10 (d) 35

COMPREHENSION

Let f be a function satisfying

$$f(x) = \frac{a}{a^x + \sqrt{a}} = g_a(x)\,(a > 0)$$

7. Let $f(x) = g_9(x)$, then the value of $\left[\displaystyle\sum_{r=1}^{1995} f\left(\frac{r}{1996}\right)\right]$ is (where [.] denotes the greatest integer function)
(a) 995 (b) 996 (c) 997 (d) 998

8. If the value of $\displaystyle\sum_{r=0}^{2n} f\left(\frac{r}{2n+1}\right) = \frac{1}{1+\sqrt{a}} + 987$, then the value of n is
(a) 493 (b) 494 (c) 987 (d) 988

INTEGER TYPE

9. The remainder when 2^{2003} is divided by 17 is

MATRIX MATCH

10. Match the following.

	List-I	List-II	
P.	$\lim_{x \to 0}\left(\dfrac{\tan x}{x}\right)^{1/x} =$	1.	e
Q.	The number of points at which $f(x) = \|\,\|x\| - 1\|$ is not differentiable, is	2.	1
R.	$\displaystyle\int_{1/e}^{\tan x}\frac{4t\,dt}{1+t^2} + \int_{1/e}^{\cot x}\frac{4\,dt}{t(1+t^2)} =$	3.	2
S.	The degree of the differential equation satisfied by all circles of radius r is	4.	3
		5.	4

	P	Q	R	S
(a)	3	2	5	4
(b)	1	2	4	3
(c)	3	2	4	5
(d)	2	4	5	3

See Solution Set of Maths Musing 176 on page no 85

JEE WORK OUTS

TWO DIMENSIONAL GEOMETRY

Time : 1 hr. **Marks :** 60

MULTIPLE CORRECT CHOICE TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) out of which ONE or MORE may be correct. [Correct ans. 3 marks & wrong ans., no negative mark]

1. If e_1 and e_2 are the eccentricities of the conic sections $16x^2 + 9y^2 = 144$ and $9x^2 - 16y^2 = 144$, then
(a) $e_1^2 + e_2^2 = 3$ (b) $e_1^2 + e_2^2 > 3$
(c) $e_1^2 + e_2^2 < 3$ (d) $e_1^2 - e_2^2 < 0$

2. The equation(s) to the tangent(s) to the conic $x^2 + 4xy + 3y^2 - 5x - 6y + 3 = 0$, which are parallel to $x + 4y = 0$ are
(a) $x + 4y - 1 = 0$ (b) $x + 4y - 3 = 0$
(c) $x + 4y - 5 = 0$ (d) $x + 4y - 8 = 0$

3. Consider the parabola $y^2 = 4ax$ and $x^2 = 4by$. The straight line $b^{1/3}y + a^{1/3}x + a^{2/3}b^{2/3} = 0$
(a) touches $y^2 = 4ax$
(b) touches $x^2 = 4by$
(c) intersects both parabolas in real points
(d) touches first and intersects other

4. The co-ordinates of a point on the parabola $y^2 = 8x$ whose distance from the circle $x^2 + (y + 6)^2 = 1$ is minimum is
(a) (2, 4) (b) (2, –4)
(c) (18, –12) (d) 8, 8

5. The angle between the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is
(a) $\cos^{-1}\left(\frac{1}{e}\right)$ (b) $2\cos^{-1}\left(\frac{1}{e}\right)$
(c) $\sin^{-1}\left(\frac{1}{e}\right)$ (d) None of these

6. The circle $x^2 + y^2 + 4x - 6y + 3 = 0$ is one of the circles of a coaxial system of circles having as radical axis the line $2x - 4y + 1 = 0$. Then the equation of the circle of the system which touches the line $x + 3y - 2 = 0$ is
(a) $x^2 + y^2 + 2x - 2y + 2 = 0$
(b) $x^2 + y^2 + 2x + 6y = 0$
(c) $x^2 + y^2 - 2x + 6y = 0$
(d) $x^2 + y^2 + 2x - 6y = 0$

7. If a circle of constant radius $3k$ passes through the origin and meets the axes at 'A' and 'B', the locus of the centroid of ΔOAB is
(a) $x^2 + y^2 = k^2$ (b) $x^2 + y^2 = 2k^2$
(c) $x^2 + y^2 = 3k^2$ (d) None of these

8. $\frac{x^2}{P^2 - P - 6} + \frac{y^2}{P^2 - 6P + 5} = 1$ will represent the ellipse if P lies in the interval
(a) $(-\infty, -2)$ (b) $(1, \infty)$
(c) $(3, \infty)$ (d) $(5, \infty)$

9. If the eccentric angles of the extremities of a focal chord of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are α and β, then
(a) $e = \frac{\cos\alpha + \cos\beta}{\cos(\alpha + \beta)}$ (b) $e = \frac{\sin\alpha + \sin\beta}{\sin(\alpha + \beta)}$

By : **Vidyalankar Institute**, Pearl Centre, Senapati Bapat Marg, Dadar (W), Mumbai - 28. Tel.: (022) 24306367

(c) $\cos\left(\frac{\alpha-\beta}{2}\right)=e\cos\left(\frac{\alpha+\beta}{2}\right)$

(d) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}$

10. If b and c are the lengths of the segments of any focal chord of a parabola $y^2 = 4ax$, then the length of the semi-latus rectum is

(a) $\frac{b+c}{2}$ (b) $\frac{bc}{b+c}$

(c) $\frac{2bc}{b+c}$ (d) $\sqrt{bc}$

ONE INTEGER VALUE CORRECT TYPE

This section contains 10 questions. Each question, when worked out will result in one integer from 0 to 9 (both inclusive). [Correct ans. 3 marks & wrong ans., no negative mark]

11. The locus of the centre of the circle for which one end of diameter is (3, 3) while the other end lies on the line $x + y = 4$ is $x + y = k$, then k equals_____

12. The greatest distance of the point (10, 7) from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is 5α, then α equals____

13. Angle between the tangents drawn from (1, 4) to the parabola $y^2 = 4x$ is $\frac{\pi}{m}$, where m equals_____

14. If the straight line $y = 2x + c$ is a tangent to the ellipse $\frac{x^2}{8}+\frac{y^2}{4}=1$, then $|c|$ equals_____

15. The foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide, then value of b^2 equals_____

16. If the angle between the two lines represented by $2x^2+ 5xy +3y^2+ 7y + 4 = 0$ is $\tan^{-1}(m)$, then the value of 10 m must be_____

17. From a point, common tangents are drawn to the circle $x^2 + y^2 = 8$ and parabola $y^2 = 16x$. If the area of the quadrilateral formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola is $10k$, then find k.

18. The line $x + y = a$ meets the x-axis at A and y-axis at B. A ΔAMN is inscribed in the ΔOAB, O being the origin,with right angle at N; M and N lie respectively on OB and AB. If the ratio of $\frac{\text{Area }(\Delta AMN)}{\text{Area }(\Delta OAB)}=\frac{3}{8}$, then find the value of $\frac{AN}{BN}$.

19. $A(0, 0)$, $B(4, 2)$ and $C(6, 0)$ are the vertices of a triangle ABC and BD is its altitude. The line through D parallel to the side AB intersects the side BC at a point E. Find the product of areas of ΔABC and ΔBDE.

20. Find the number of integral values of λ if $(\lambda , 2)$ is an interior point of ΔABC formed by, $x + y = 4$, $3x - 7y = 8$, $4x - y = 31$.

ANSWER KEY

1.	(c, d)	2.	(c, d)	3.	(a, b)	4.	(b)
5.	(b)	6.	(a, c)	7.	(d)	8.	(a, d)
9.	(b, c, d)	10.	(c)	11.	(5)	12.	(3)
13.	(3)	14.	(6)	15.	(7)	16.	(2)
17.	(6)	18.	(3)	19.	(8)	20.	(1)

For detailed solution to the Sample Paper, visit our website www. vidyalankar.org

JEE Main 2018

MOCK TEST PAPER

Series-3

Time: 1 hr 15 min.

The entire syllabus of Mathematics of JEE MAIN is being divided into eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given below:

Unit No. 3	Topic	Syllabus In Details
	Permutations and Combinations	Fundamental principle of counting, permutation as an arrangement and combination as selection, meaning of $P(n, r)$ and $C(n, r)$, simple applications.
	Trigonometry	General solution and Properties of Triangle.
	Co-ordinate Geometry-2D	Circles: Standard form of equation of a circle, general form of the equation of a circle, its radius and centre, equations of a circle when the end points of a diameter are given, points of intersection of a line and a circle with the centre at the origin and condition for a line to be tangent to a circle, equation of the tangent.

1. The number of positive terms in the sequence $x_n = \dfrac{195}{4 \cdot {}^nP_n} - \dfrac{{}^{n+3}P_3}{{}^{n+1}P_{n+1}}$, $n \in N$ is
(a) 2 (b) 3
(c) 4 (d) none of these.

2. The number of numbers greater than 50,000 that can be formed by using the digits 3, 5, 6, 6, 7 is
(a) 36 (b) 48
(c) 54 (d) none of these.

3. The rank of the word 'FLOWER' is
(a) 165 (b) 155
(c) 145 (d) none of these.

4. A person write letters to his 4 friends and addressed the corresponding envelopes. The number of ways at least two of them are in the wrong envelopes is
(a) 23 (b) 19 (c) 17 (d) 14.

5. A test consists of 6 multiple choice questions each having 4 alternative answers of which only one is correct. Also only one of the alternatives must be marked by each candidate. The number of ways of getting exactly 4 correct answers by a candidate answering all the questions is
(a) $4^6 - 3^2$ (b) 135
(c) 55 (d) 120.

6. Number of ways in which 3 boys and 3 girls(all are of different heights) can be arranged in a line so that boys as well as girls among themselves are in decreasing order to their height (from left to right) is
(a) 720 (b) 72
(c) 10 (d) 20.

7. The number of permutations of the letters of the word HINDUSTAN such that neither the pattern 'HIN' nor 'DUS' nor 'TAN' appears, are
(a) 166674 (b) 169194
(c) 166680 (d) 181434.

8. The number of ways in which four letters of the word MATHEMATICS can be arranged is given by
(a) $\dfrac{8!}{4!4!}$ (b) 2454
(c) 2464 (d) 2474.

By : Sankar Ghosh, S.G.M.C, Mob : 09831244397.

9. The number of ways of distributing of 50 identical things among 8 persons in such a way that three of them get 8 things each, two of them get 7 things each, and remaining three of them get 4 things each, is equal to

(a) $\dfrac{(50!)(8!)}{(8!)^3(3!)^2(7!)^2(4!)^3(2!)}$

(b) $\dfrac{(50!)(8!)}{(8!)^3(7!)^3(4!)^3}$

(c) $\dfrac{(50!)}{(8!)^3(7!)^2(4!)^3}$ (d) $\dfrac{8!}{(3!)^2\cdot 2!3!}\cdot 1$

10. Consider seven digit number x_1, x_2, ..., x_7 where $x_1, x_2, ..., x_7 \neq 0$ having the property that x_4 is the greatest digit and digits towards the left and right of x_4 are in decreasing order. Then total number of such numbers in which all digits are distinct is

(a) ${}^9C_7 \cdot {}^6C_3$ (b) ${}^9C_6 \cdot {}^5C_3$

(c) ${}^{10}C_7 \cdot {}^6C_3$ (d) none of these.

11. The most general solution of the equation $\log_{\cos\theta}\tan\theta + \log_{\sin\theta}\cot\theta = 0$, is

(a) $n\pi + \dfrac{\pi}{4}$ (b) $n\pi - \dfrac{\pi}{4}$

(c) $2n\pi - \dfrac{\pi}{4}$ (d) $2n\pi + \dfrac{\pi}{4}$

12. The general solution of $2 - \cos x = 2\tan\dfrac{x}{2}$ is

(a) $(2n+1)\dfrac{\pi}{2}$ (b) $(4n+1)\dfrac{\pi}{2}$

(c) $2n\pi$ (d) $(4n + 1)\pi$.

13. The general solution of

$\sin x + \cos x = \min\limits_{a\in R}\{1, a^2 - 4a + 6\}$ is

(a) $\dfrac{n\pi}{2} + (-1)^n\dfrac{\pi}{4}$ (b) $2n\pi + (-1)^n\dfrac{\pi}{4}$

(c) $n\pi + (-1)^{n+1}\dfrac{\pi}{4}$ (d) $n\pi + (-1)^n\dfrac{\pi}{4} - \dfrac{\pi}{4}$

14. The equation $\cos^4\theta + \sin^4\theta + \lambda = 0$ has real solution for θ, if

(a) $\dfrac{3}{4} \le \lambda \le 1$ (b) $-1 \le \lambda \le -\dfrac{1}{2}$

(c) $0 \le \lambda \le 1$ d) $\lambda < -1$

15. In a triangle ABC, if $\angle B = \dfrac{\pi}{3}$, $\angle C = \dfrac{\pi}{4}$ and D divides BC internally in the ratio 1:3, then $\dfrac{\sin\angle BAD}{\sin\angle CAD}$ equals

(a) $\dfrac{1}{3}$ (b) $\dfrac{1}{\sqrt{3}}$ (c) $\dfrac{1}{\sqrt{6}}$ (d) $\sqrt{\dfrac{2}{3}}$

16. The ratio of the sides of a triangle is 19:16:5, then $\cot\dfrac{A}{2} : \cot\dfrac{B}{2} : \cot\dfrac{C}{2}$ equals

(a) 1:15:4 (b) 15:1:4

(c) 4:1:15 (d) 1:4:15.

17. If the sides a, b, c of a triangle ABC are the roots of the equation $x^3 - 15x^2 + 47x - 82 = 0$ then the value of $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c}$ equals

(a) $\dfrac{225}{164}$ (b) $\dfrac{131}{164}$ (c) $\dfrac{131}{82}$ (d) $\dfrac{169}{82}$

18. If two sides of a triangle are $2\sqrt{3} - 2$ and $2\sqrt{3} + 2$ and their included angle is 60°, then the other angles are

(a) 75°, 45° (b) 105°, 15°

(c) 60°, 60° (d) 90°, 30°

19. The maximum value of $\dfrac{a\cos A + b\cos B + c\cos C}{a + b + c}$ is

(a) $\sin 2A$ (b) $4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

(c) $\sin\dfrac{C}{2}$ (d) none of these.

20. Let a, b, c are the sides of a triangle and $(\sin A + \sin B + \sin C)(\sin A + \sin C - \sin B) = \mu \sin A \sin C$ where $\sin A = ak$, $\sin B = bk$, $\sin C = ck$ then the range of μ is

(a) [0, 1] (b) [–4, 4]

(c) (0, 4) (d) [0, 4]

21. A line passing through the point (11, –2) and touching the circle $x^2 + y^2 = 25$ are

(a) $4x + 3y = 38$, $7x - 24y = 125$

(b) $3x + 4y = 25$, $7x - 24y = 125$

(c) $3x - 4y = 41$, $7x + 24y = 125$

(d) $7x - 24y = 125$, $4x - 3y = 38$

22. Two circles each of radius 5, have a common tangent at (1, 1) whose equation is $4x + 3y - 7 = 0$ The centres are

(a) (–4, 4), (6, 2) (b) (–3, 4), (5, –2)

(c) (5, 4), (–3, –2) (d) (4, 2), (–2, 0)

23. The equation of the circles touching the line $x + 2y = 0$ and passing through the points of intersection of the circle $x^2 + y^2 = 4$ and $x^2 + y^2 - 2x - 4y + 4 = 0$ is
(a) $x^2 + y^2 + x + 2y = 0$
(b) $x^2 + y^2 - x + 2y = 0$
(c) $x^2 + y^2 + x - 2y = 0$
(d) $x^2 + y^2 - x - 2y = 0$.

24. A circle which passes through the point (1, 1) and cuts orthogonally the two circles $x^2 + y^2 - 8x - 2y + 16 = 0$ and $x^2 + y^2 - 4x - 4y + 1 = 0$. If its centre (a, b), then $a + b =$

(a) 0 (b) -1 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

25. The $x + y + 1 = 0$ meets the circle $x^2 + y^2 + 3x + y - 6 = 0$ at the points A and B. If C is a point on the circle, then the locus of the orthocentre of the triangle ABC is
(a) $x^2 + y^2 - x + y - 8 = 0$
(b) $x^2 + y^2 - x + y = 0$
(c) $x^2 + y^2 + x - y - 8 = 0$
(d) $x^2 + x - y = 0$

26. The circle C_1 touches the line $x - y = 0$ at the origin and the circle C_2 touches the line $x + y = 0$ at the origin. Let C be the circle $x^2 + y^2 - 4x + 2y - 6 = 0$. The common chord of C and C_1 pass through A and common chords of C and C_2 pass through B. Then $AB =$

(a) 5 (b) $\sqrt{5}$ (c) $2\sqrt{5}$ (d) $3\sqrt{5}$

27. The triangle PQR is inscribed in the circle $x^2 + y^2 = 25$ if Q and R have co-ordinates (3, 4) and (–4, 3), then $\angle QPR$ is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ d) $\frac{\pi}{6}$

28. In an equilateral triangle, 3coins of radii 1 are kept so that they touch each other and also the sides of the triangle. Area of the triangle is

(a) $4+2\sqrt{3}$ (b) $6+4\sqrt{3}$

(c) $12+\frac{7\sqrt{3}}{4}$ (d) $3+\frac{7\sqrt{3}}{4}$

29. Consider a family of circles which are passing through the point $(-1, 1)$ and are tangent to x-axis. If (h, k) is the centre of circle, then

(a) $k \geq \frac{1}{2}$ (b) $-\frac{1}{2} \leq k \leq \frac{1}{2}$

(c) $k \leq \frac{1}{2}$ (d) $0 < k < \frac{1}{2}$

30. If (–4, 3) and (12, –1) are the ends of diameter of a circle which makes an intercept of 2λ on the y-axis, then $\lambda =$

(a) $\sqrt{13}$ (b) $4\sqrt{13}$ (c) $3\sqrt{13}$ (d) $2\sqrt{13}$

SOLUTIONS

1. (c): Given that $x_n = \frac{195}{4\cdot{}^nP_n} - \frac{{}^{n+3}P_3}{{}^{n+1}P_{n+1}},\ n \in N$

$$= \frac{195}{4\cdot n!} - \frac{(n+2)(n+1)}{(n+1)!} = \frac{195}{4\cdot n!} - \frac{(n+3)(n+2)}{n!}$$

$$= \frac{195-4n^2-20n-24}{4n!} = \frac{171-4n^2-20n}{4\cdot n!}$$

$\because$ x_n is positive $\therefore$ $\frac{171-4n^2-20n}{4\cdot n!} > 0$

$\Rightarrow\ 4n^2+20n-171<0$

The above inequality is true for $n = 1, 2, 3$ and 4.

2. (b) : Here, we have 5 digits among which 6 is repeated twice and number greater than 50,000 are consists of 5 digits.

$\therefore$ Number of permutation $= \frac{5!}{2!} = \frac{120}{2} = 60$

But there are some numbers among these 60 numbers which are started with 3 and those are less than 50,000, so they are to be rejected.

Such number of numbers $= \frac{4!}{2!} = 12$

$\therefore$ The required number of numbers = 60 – 12 = 48

3. (b) : Words before FLOWER are
$1 \times 5! + 1 \times 4! + 1 \times 3! + 2 \times 2! = 154$

$\therefore$ Rank of the word FLOWER is 154 + 1 = 155.

4. (a) : Let D_r denotes the number of ways in which r things goes to wrong places. If r goes to wrong place out of n then $(n - r)$ goes to correct places.

$\therefore$ $D_n = {}^nC_{n-r}D_r$

where $D_r = r!\left(\frac{1}{2}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^r\frac{1}{r!}\right)$

$$\therefore \sum_{i=1}^{r} D_n = \sum_{i=1}^{r} {}^nC_{n-r}D_i \quad (r<n)$$

Now for our problem

$$(D_r)_{r=4} = 4!\left(\frac{1}{2}-\frac{1}{3!}+\frac{1}{4!}\right) = 12-4+1 = 9 = D_4$$

$$(D_r)_{r=3} = 3!\left(\frac{1}{2}-\frac{1}{3!}\right) = 3-1 = 2 = D_3$$

$$(D_r)_{r=2} = 2!\left(\frac{1}{2!}\right) = 1 = D_2$$

Now no. of ways at least two goes to the wrong places, $n = 4, r \geq 2$

$$\sum_{i=2}^{4} {}^nC_{n-r}D_r = {}^4C_2D_2 + {}^4C_1D_3 + {}^4C_0D_4$$

$= 6D_2 + 4D_3 + D_4 = 6 + 8 + 9 = 23$

So, 23 are the number of ways in which at least two things goes to wrong place.

5. (b) : The number of ways by which 4 questions with correct answers can be chosen in ${}^6C_4$. Since only one of the four alternatives is correct, the wrong answers can be given in 3 different ways for each of the two remaining questions attempted unsuccessfully by the candidate. The desired number of ways $= {}^6C_4 \cdot 3^2 = 135$

6. (d) : Since order of boys and girls are to be maintained in any of the different arrangements, the required number $= \dfrac{6!}{3!3!}$

7. (b) : Total number of permutations $= \dfrac{9!}{2!}$; no. of permutations where 'HIN' are always together = 7!; no. of permutations where 'DUS' are always together $= \dfrac{7!}{2!}$ and no. of permutations where 'TAN' are always together= 7!;

Now the number of permutations where 'HIN' and 'DUS' are always together = 5!

Number of permutations where 'HIN' and 'TAN' are always together = 5!

Number of permutations where 'TAN' and 'DUS' are always together = 5!

Number of permutations where 'HIN', 'DUS' and 'TAN' are always together = 3!

Required number of permutations

$$= \frac{9!}{2!} - \left(7! + 7! + \frac{7!}{2!}\right) + 3 \times 5! - 3! = 169194$$

8. (b) : The word 'MATHEMATICS' consists of 11 letters in which there are 2M's, 2A's and 2 T's

The following cases are possible.

Case I: Four letters having a pair of similar letters. This can be done in ${}^3C_2$ ways and no. of permutation of such four letters is $\dfrac{4!}{2!2!}$

Case II: one similar pair and 2 different letters. This can be done in ${}^3C_1 \times {}^7C_2$ ways and then no. of possible permutations $= \dfrac{4!}{2!}$

Case III: All different letters are taken. This can be done in ${}^8C_4$ ways and then no. of permutation = 4!

Therefore the required no. is

$${}^3C_2 \times \frac{4!}{2!2!} + {}^3C_1 \times {}^7C_2 \times \frac{4!}{2!} + {}^8C_4 \times 4! = 2454.$$

9. (d) : Number of ways of dividing 8 persons in three groups first having 3 persons, second having 2 persons and third having 3 persons $= \dfrac{8!}{3!2!3!}$

Since all the 50 things are identical

$\therefore$ Required number $= \dfrac{8!}{(3!)^2 \cdot 2!3!} \cdot 1$

10. (a) : Number of selections of 7 digits out of digits 1, 2, 3, ..., 9 $= {}^9C_7$

Out of these 7 selected digits excluding the greatest digit = 6, the 6 digits can be divided in two groups each having 3 digits $= \dfrac{6!}{3! \times 3! \times 2!} = {}^6C_3 \times \dfrac{1}{2!}$

But the three digits on one side can go on the other side

$\therefore$ Required number $= {}^9C_7 \cdot {}^6C_3 \dfrac{1}{2!} .2! = {}^9C_7 \cdot {}^6C_3$

11. (a) : The given equation is

$\log\cos\theta\tan\theta + \log\sin\theta\cot\theta = 0$

$$\Rightarrow \log_{\cos\theta} \tan\theta - \log_{\sin\theta} \tan\theta = 0$$

$$\Rightarrow \frac{\log \tan\theta}{\log\cos\theta} = \frac{\log\tan\theta}{\log\sin\theta} \Rightarrow \frac{\log\sin\theta}{\log\cos\theta} = 1$$

$$\Rightarrow \log_{\cos\theta} \sin\theta = 1 \Rightarrow \cos\theta = \sin\theta \Rightarrow \tan\theta = \tan\frac{\pi}{4}$$

$$\therefore \theta = n\pi + \frac{\pi}{4}, \; n \in Z.$$

12. (b) : The given equation is $2 - \cos x = 2\tan\dfrac{x}{2}$

$$\Rightarrow 2\left(1 - \tan\frac{x}{2}\right) = \cos x \Rightarrow 2\left(1 - \tan\frac{x}{2}\right) = \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$$

$$\Rightarrow \left(1 - \tan\frac{x}{2}\right)\left(2 - \frac{1 + \tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right) = 0$$

Either $1 - \tan\dfrac{x}{2} = 0$ or $2 - \dfrac{1 + \tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = 0$

$$\Rightarrow \tan\frac{x}{2} = 1 \text{ or } 2\tan^2\frac{x}{2} - \tan\frac{x}{2} + 1 = 0$$

$$\Rightarrow \tan\frac{x}{2} = 1 = \tan\frac{\pi}{4} \Rightarrow \frac{x}{2} = n\pi + \frac{\pi}{4}$$

$$\Rightarrow x = 2n\pi + \frac{\pi}{2} \Rightarrow x = (4n+1)\frac{\pi}{2}$$

and $2\tan^2\dfrac{x}{2} - \tan\dfrac{x}{2} + 1 = 0$

$\Rightarrow \quad \tan\frac{x}{2}=\frac{1\pm\sqrt{1^2-4\cdot 2\cdot 1}}{2\cdot 2}=\frac{1\pm i\sqrt{7}}{4}$ (no solution)

13. (d) : Here, $a^2 - 4a + 6 = (a - 2)^2 + 2 \geq 2$

$\therefore \min\limits_{a\in R}\{1, a^2-4a+6\}=1$

Now, $\sin x+\cos x=1\Rightarrow \frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x=\frac{1}{\sqrt{2}}$

$\Rightarrow \sin\left(x+\frac{\pi}{4}\right)=\sin\frac{\pi}{4} \quad \therefore \quad x+\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{4}$

$\therefore \quad x=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4}$

14. (b) : We have, $\cos^4\theta + \sin^4\theta + \lambda = 0$

$\Rightarrow 1-2\sin^2\theta\cos^2\theta=-\lambda$

$\Rightarrow 1-\frac{1}{2}\sin^2 2\theta=-\lambda \Rightarrow 1-\frac{1}{4}(1-\cos 4\theta)=-\lambda$

$\Rightarrow \frac{3}{4}+\frac{1}{4}\cos 4\theta=-\lambda$

$\because -1\leq\cos 4\theta\leq 1 \;\therefore\; -\frac{1}{4}\leq\frac{1}{4}\cos 4\theta\leq\frac{1}{4}$

$\Rightarrow \frac{3}{4}-\frac{1}{4}\leq\frac{3}{4}+\frac{1}{4}\cos 4\theta\leq\frac{3}{4}+\frac{1}{4}$

$\Rightarrow \frac{1}{2}\leq-\lambda\leq 1 \Rightarrow -1\leq\lambda\leq-\frac{1}{2}$

15. (c): From ΔABD we have, $\frac{BD}{\sin\angle BAD}=\frac{AD}{\sin B}$

From ΔACD we have, $\frac{CD}{\sin\angle CAD}=\frac{AD}{\sin C}$.

$\because BD:CD=1:3 \;\therefore\; \frac{AD\sin\angle BAD}{\sin B}:\frac{AD\sin\angle CAD}{\sin C}=1:3$

$\Rightarrow \frac{\sin\angle BAD}{\sin\frac{\pi}{3}}:\frac{\sin\angle CAD}{\sin\frac{\pi}{4}}=1:3$

$\Rightarrow \frac{\sqrt{2}\sin\angle BAD}{\sqrt{3}\sin\angle CAD}=\frac{1}{3} \Rightarrow \frac{\sin\angle BAD}{\sin\angle CAD}=\frac{1}{\sqrt{6}}$

16. (d) : Given $a : b : c = 19 : 16 : 5$

$\therefore \quad 2s = a + b + c = 40$

$\therefore \quad \cot\frac{A}{2}:\cot\frac{B}{2}:\cos\frac{C}{2}$

$=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}:\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}:\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$= (s - a) : (s - b) : (s - c)$

$=\left(\frac{a+b+c}{2}-a\right):\left(\frac{a+b+c}{2}-b\right):\left(\frac{a+b+c}{2}-c\right)$

$=\left(\frac{40}{2}-19\right):\left(\frac{40}{2}-16\right):\left(\frac{40}{2}-15\right)=1:4:15$

17. (b) : We are given that, a, b, c are the roots of the equation $x^3 - 15x^2 + 47x - 82 = 0$

$\therefore \quad a + b + c = 15,\ ab + bc + ca = 47,\ abc = 82$

Now $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$

$=\frac{b^2+c^2-a^2}{2abc}+\frac{c^2+a^2-b^2}{2abc}+\frac{a^2+b^2-c^2}{2abc}=\frac{a^2+b^2+c^2}{2abc}$

$=\frac{(a+b+c)^2-2(ab+bc+ca)}{2abc}=\frac{(15)^2-2(47)}{2(82)}=\frac{131}{164}$

18. (b) : Let $b=2\sqrt{3}+2,\ c=2\sqrt{3}-2$ and $A=60°$

$\therefore \tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{4}{4\sqrt{3}}\cot 30°=1=\tan 45°$

$\Rightarrow B-C=90°$ Also, $B+C=120°$ $(\because A=60°)$

Therefore the other two angles are $B = 105°$ and $C = 15°$

19. (b) : $\frac{a\cos A+b\cos B+c\cos C}{a+b+c}$

$=\frac{2\sin A\cos A+2\sin B\cos B+2\sin C\cos C}{2(\sin A+\sin B+\sin C)}$

$\left[\because \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R\right]$

$=\frac{\sin 2A+\sin 2B+\sin 2C}{2(\sin A+\sin B+\sin C)}$

$=\frac{4\sin A\sin B\sin C}{2\left(4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)}=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

20. (c) : $(\sin A + \sin B + \sin C)(\sin A + \sin C - \sin B)$
$= \mu\sin A\sin C$ (given)

$(a + b + c)(a + c - b) = \mu ac$

$\left(\because \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R\right)$

$\Rightarrow \quad (a + c)^2 - b^2 = \mu ac \Rightarrow a^2 + c^2 - b^2 + 2ac = \mu ac$

$\Rightarrow \quad \frac{a^2+c^2-b^2}{2ac}+1=\frac{\mu}{2}$

$\Rightarrow \quad 1+\cos B=\frac{\mu}{2} \quad \left(\because \cos B=\frac{a^2+c^2-b^2}{2ac}\right)$

$\Rightarrow \quad \cos^2\frac{B}{2}=\frac{\mu}{4} \Rightarrow 0<\frac{\mu}{4}<1 \ (\because \angle B\neq 0 \text{ and } \angle B\neq 180°)$

$\therefore \quad 0 < \mu < 4$

21. (b) : Let the equation of the line be

$y + 2 = m(x - 11)$ or $mx - y - (11m + 2) = 0$ ...(i)

Since the line (i) touches the circle $x^2 + y^2 = 25$

$\therefore \quad \frac{|-(11m+2)|}{\sqrt{m^2+1}}=5\Rightarrow m=-\frac{3}{4},\frac{7}{24}$

Therefore (i) becomes $3x + 4y = 25$ or $7x - 24y = 125$.

22. (c) : The line joining the centres passing through (1, 1) and perpendicular to the tangent $4x + 3y - 7 = 0$ is
$3x - 4y + 1 = 0$ ...(i)
Let the abscissa of the centre be α and its ordinate can be obtained from (i)

$\therefore$ Centre $C\left(\alpha, \frac{3\alpha+1}{4}\right)$

The distance between the centre C and (1, 1) is 5

$$\therefore \quad (\alpha-1)^2 + \left(\frac{3\alpha+1}{4}-1\right)^2 = 25$$

On solving the above equation we get $\alpha = 5, -3$ therefore $C = (5, 4), (-3, -2)$

23. (d) : The equation of the circle passing through the intersection of the circle
$x^2 + y^2 = 4$ and $x^2 + y^2 - 2x - 4y + 4 = 0$ is
$(x^2 + y^2 - 2x - 4y + 4) + \lambda(x^2 + y^2 - 4) = 0$

$$\Rightarrow (1+\lambda)x^2 + (1+\lambda)y^2 - 2x - 4y + 4(1-\lambda) = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{2x}{1+\lambda} - \frac{4y}{1+\lambda} + \frac{4(1-\lambda)}{1+\lambda} = 0 \quad \text{...(i)}$$

Since the line $x + 2y = 0$ touches the circle (i)

$$\therefore \quad \frac{\frac{1}{1+\lambda}+\frac{4}{1+\lambda}}{\sqrt{1^2+2^2}} = \sqrt{\left(\frac{1}{1+\lambda}\right)^2 + \left(\frac{2}{1+\lambda}\right)^2 - \frac{4(1-\lambda)}{1+\lambda}}$$

$$\Rightarrow \quad \frac{\frac{5}{1+\lambda}}{\sqrt{5}} = \sqrt{\frac{5-(4)(1-\lambda^2)}{(1+\lambda)^2}}$$

$$\Rightarrow \quad \frac{\sqrt{5}}{1+\lambda} = \sqrt{\frac{1+4\lambda^2}{(1+\lambda)^2}} \quad \Rightarrow \quad \frac{5}{(1+\lambda)^2} = \frac{1+4\lambda^2}{(1+\lambda)^2}$$

$$\Rightarrow \quad 1+4\lambda^2 = 5 \quad \Rightarrow \quad \lambda^2 = 1 \Rightarrow \lambda = 1$$

Now put the value of $\lambda = 1$ in equation (i) we get $x^2 + y^2 - x - 2y = 0$

24. (d) : Let the equation of the circle be
$x^2 + y^2 + 2gx + 2fy + c = 0.$
It cuts the two given circles orthogonally
$\therefore \quad -8g - 2f = c + 16$ and $-4g - 4f = c + 1$
$\Rightarrow \quad 8g + 2f + c = -16$ ...(i)

and $4g + 4f + c = -1$ ...(ii)
The circle passes through (1, 1)
$\therefore \quad 2g + 2f + c = -2$ ...(iii)

Solving (i), (ii) and (iii) we get $g = -\frac{7}{3}, f = \frac{17}{6}, c = -3$

$\therefore$ The centre is $\left(\frac{7}{3}, -\frac{17}{6}\right) = (a, b) \Rightarrow a = \frac{7}{3}, b = -\frac{17}{6}, a+b = -\frac{1}{2}$

25. (c) : Let (h, k) be the orthocentre. The image of (h, k) on the line $x + y + 1 = 0$ is given by

$$\frac{x-h}{1} = \frac{y-k}{1} = \frac{-2(h+k+1)}{1+1}$$

The image is $(-1 - k, -1 - h)$
It lies on the circumcircle of triangle ABC.
$\therefore \ (1 + k)^2 + (1 + h)^2 - 3(1 + k) - (1 + h) - 6 = 0$

$$\Rightarrow \quad h^2 + k^2 + h - k - 8 = 0$$

Therefore the locus is $x^2 + y^2 + x - y - 8 = 0$

26. (c) : Here $C_1 = x^2 + y^2 + \lambda(x - y) = 0, \lambda = R,$
$C_2 : x^2 + y^2 + \lambda(x+y) = 0, \quad \lambda \in R$
and $C : x^2 + y^2 - 4x + 2y - 6 = 0$
The common chord of C and C_1 are
$\lambda(x - y) + 4x - 2y + 6 = 0$ which are concurrent at
$x - y = 0, \Rightarrow 4x - 2x + 6 = 0 \Rightarrow x = -3 \ \therefore \ A = (-3, -3)$
The common chord of C and C_2 are
$\lambda(x + y) + 4x - 2y + 6 = 0$
$\Rightarrow x = -y, \ -6y + 6 = 0 \Rightarrow y = 1 \ \therefore \ B = (-1, 1)$

$$\therefore \quad AB = \sqrt{4^2 + 2^2} = 2\sqrt{5}$$

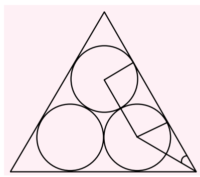
27. (c) : The centre of the circle $x^2 + y^2 = 25$ is (0, 0).
Now slope of $QO \times$ slope of RO

$$= \frac{4-0}{3-0} \times \frac{3-0}{-4-0} = \frac{4}{3} \times -\frac{3}{4} = -1$$

Therefore angle at the centre is $\angle QOR = \frac{\pi}{2}$,

$$\therefore \quad \text{Angle at circumference } \angle QPR = \frac{1}{2} \times \angle QOR = \frac{\pi}{4}$$

28. (b) : Length of the side



$$= 2 + 2\cot\frac{\pi}{6} = 2(1+\sqrt{3})$$

$$\text{Area} = \frac{\sqrt{3}}{4}\left(2\left(1+\sqrt{3}\right)\right)^2 = 6 + 4\sqrt{3}$$

29. (a) : Circle with centre (h, k) and touching x-axis is
$x^2 + y^2 - 2hx - 2ky + h^2 = 0$ ....(i)

$\because$ (−1,1) lies on (i) $\therefore \ 2 + 2h - 2k + h^2 = 0$

$$\Rightarrow \quad (h^2 + 2h + 1) + 1 - 2k = 0 \ \Rightarrow \ 2k = (h+1)^2 + 1$$

$$\Rightarrow k = \frac{1}{2} + \frac{1}{2}(h+1)^2 \quad \Rightarrow \quad k \geq \frac{1}{2}$$

30. (d) : The equation of circle having extremities of the diameter (−4, 3) and (12, −1) is
$(x + 4)(x - 12) + (y - 3)(y + 1) = 0$

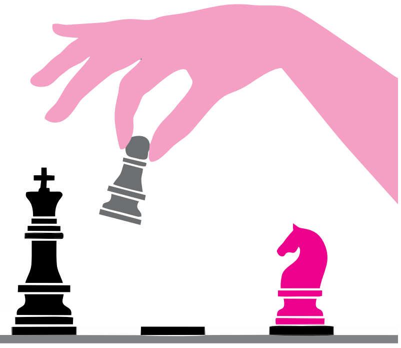
$$\Rightarrow \quad x^2 + y^2 - 8x - 2y - 51 = 0$$

Now, length of the chord intercepted on y-axis is

$$2\sqrt{f^2 - c} = 2\sqrt{(-1)^2 - (-51)} = 2\sqrt{52} = 4\sqrt{13}$$

$$\therefore \quad 2\lambda = 4\sqrt{13} \ \Rightarrow \ \lambda = 2\sqrt{13}$$

Challenging PROBLEMS ON CALCULUS



1. Consider a function $f:(0,\infty)\to R$ and a real number $a > 0$ such that $f(a) = 1$ and
$$f(x)\cdot f(y)+f\left(\frac{a}{x}\right)\cdot f\left(\frac{a}{y}\right)=2f(xy)$$
for all $x, y\in(0, \infty)$ and $f(2) = 1$ then $f(3) =$
(a) 1 (b) 6 (c) 9 (d) 16

2. Consider the polynomial $P(x)$ with integral coefficients such that $P(P'(x))=P'(P(x))$ and $P(4) = 4$ then $P(2) =$
(a) 2 (b) 3 (c) 4 (d) 5

3. All polynomials P of degree n having only real zeroes $x_1, x_2, x_3, ..., x_n$ such that $\sum_{i=1}^{n}\frac{1}{P(x)-x_i}=\frac{n^2}{xp'(x)}$ for all non-zero real numbers x is of the form $P(x) =$
(a) $2x^n + 3x^{n-1}$ (b) $2x^n + 3x^{n-2}$
(c) $2x^n$ (d) $2x^{n-1} + 3x^{n-2} + 4x^n$

4. Consider the polynomial with real coefficients $P(x) = a_0x^n + a_1x^{n-1} + ... + a_n, a_n \neq 0$. If the equation $P(x) = 0$ has all its roots real and distinct then the equation $x^2P''(x)+3xP'(x)+P(x)=0$ has
(a) real and distinct roots
(b) real and equal roots
(c) not real roots
(d) not all real roots

5. Let $f, g : R \to R$ be periodic functions of periods 'a' and 'b' such that $\lim_{x\to 0}\frac{f(x)}{x}=l$ and $\lim_{x\to 0}\frac{g(x)}{x}=m$, $l\in R,\ m\in R-\{0\}$. Then $\lim_{n\to\infty}\frac{f((3+\sqrt{7})^n a)}{g((2+\sqrt{2})^n b)}=$
(a) 0 (b) 1
(c) -1 (d) does not exist

6. Let $f, g : R \to R$ be two periodic functions with respective periods T_1 and T_2 and $\lim_{x\to\infty}(f(x)-g(x))=0$ then
(a) $T_1 = T_2$ (b) $T_1 = 2T_2$
(c) $2T_1 = T_2$ (d) $T_1 = 1 + T_2$

7. Let $f:[1, \infty]\to R$ be the function defined by $f(x)=\frac{\sqrt{[x]}+\sqrt{\{x\}}}{\sqrt{x}}$. The smallest number k such that $f(x) \leq k$ for all $x \geq 1$ is
(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

8. The real parameter m such that the graph of the function $f(x)=\sqrt[3]{8x^3+mx^2}-nx$ has the horizontal asymptote $y = 1$ is
(a) 10 (b) 12 (c) 14 (d) 16

9. f is continuous function $R \to R$, $f(0) = 1$ and $f(2x) - f(x) = x\ \forall\ x \in R$ then $f(2) =$
(a) 1 (b) 2 (c) 3 (d) 4

10. Real functions $f, g, h : R \to R$ are such that for all real x, y, $(x - y)f(x) + h(x) - xy + y^2 \leq h(y) \leq (x - y) g(x) + h(x) - xy + y^2$ then $h(x)$ is
(a) a linear function
(b) a quadratic function
(c) a reciprocal function
(d) an exponential function

11. Let $f(x)$ be a function which contains element 2 in its domain and range. Suppose that $f(f(x)) \cdot (1 + f(x)) = -f(x)$ for all numbers x in the domain of f, then $f(2) =$
(a) 1 (b) 2/3
(c) -1 (d) $-2/3$

By : Tapas Kr. Yogi, Visakhapatnam Mob : 09533632105

12. Let $a_0 \in (-1, 1)$ and defined recursively, $a_n = \sqrt{\frac{1+a_{n-1}}{2}}, n > 0$. Let $A_n = 4^n(1 - a_n)$ then $\lim_{n\to\infty} A_n =$

(a) $(\cos^{-1}a_0)^2$ (b) $\frac{(\cos^{-1}a_0)^2}{2}$

(c) $2(\cos^{-1}a_0)^2$ (d) $3(\cos^{-1}a_0)^2$

13. $\lim_{n\to\infty} \frac{1}{n}\cdot\sum_{k=1}^{n}\left[\frac{2n}{k}\right]-2\left[\frac{n}{k}\right] =$ ___

([·] denotes the greatest integer function)

(a) log4 (b) log4 + 1

(c) log4 – 1 (d) 2log2 – 2

14. Let $f(x)$ be a continuous function on (a, b) and $\lim_{n\to a^+} f(x) = +\infty$, $\lim_{n\to b^-} f(x) = -\infty$ and $f'(x)+f^2(x) \ge -1$ for $x \in (a, b)$ then minimum value of $(b - a)$ is

(a) π (b) $\pi/2$ (c) 2π (d) $\pi/4$

15. Let $f(x, y)$ be a function satisfying the functional equation $f(x, y) = f(2x + 2y, 2y - 2x)$ for all real number x, y. Define $g(x)$ by $g(x) = f(2^x, 0)$. Then period of $g(x)$ is

(a) 3 (b) 4 (c) 6 (d) 12

16. The number of real solutions to the equation

$\frac{x}{\sqrt{x+1}} = \frac{3}{2} + \frac{5}{16}(x-3)$ is/are

(a) 1 (b) 2 (c) 3 (d) 5

17. Consider the function,

$$f(x) = \begin{cases} \log(1+x+x^2), & x \le b \\ ax+c, & x > b \end{cases}$$

If the graph of $f(x)$ is concave in R then

(a) $b > \frac{-(1+\sqrt{3})}{2}$ (b) $c + ab = \log(b + b^2)$

(c) $a \le \frac{2b+1}{b^2+b+1}$ (d) $a > \frac{2b+1}{b^2+b+1}$

18. Consider $a_n = \sin(n\pi/2)$ and the three sequences $b_n = na_n (n \ge 0)$, $c_n = \frac{a_n}{n} (n \ge 1)$ and $d_n = a_n \cos\left(\frac{n\pi}{2}\right)$ $(n \ge 1)$ then

(a) $\lim_{n\to\infty} b_n$ does not exist

(b) $\lim_{n\to\infty} c_n$ does not exist

(c) $\lim_{n\to\infty} d_n$ does not exist

(d) $\lim_{n\to\infty} (c_n + d_n)$ does not exist

SOLUTIONS

1. (a) : Putting $x = 1 = y$ gives $f(1) = 1$

Putting $y = 1$ gives, $f(x) = f\left(\frac{a}{x}\right), x > 0$

Putting $y = \frac{a}{x}$ gives, $f(x)\cdot f\left(\frac{a}{x}\right) = 1$

so that, gives, $f^2(x) = 1$

Putting $x = y = \sqrt{t}$ gives $f^2(\sqrt{t}) + f^2\left(\frac{a}{\sqrt{t}}\right) = 2f(t)$

As L.H.S. is positive so R.H.S is positive.
Hence $f(x) = 1$ for all x.

2. (a) : Consider a polynomial

$P(x) = a_0x^n + a_1x^{n-1} + ... + a_n, a_0 \ne 0$

where $a_0, a_1, a_2, ..., a_n \in I$, then

$P'(x) = na_0x^{n-1} + (n-1)a_1x^{n-2} + ... + a_{n-1}$

By comparing coefficient of $x^{n(n-1)}$ in the relation

$P(P'(x)) = P'(P(x))$, we have $a_0^{n+1}\cdot n^n = a_0^n \cdot n$

i.e., $a_0 = \frac{1}{n^{n-1}}$.

Now, since a_0 is an integer $\Rightarrow$ $n = 1$ and $a_0 = 1$
So, $P(x) = x + a_1$
Putting back this value of $P(x)$ in the given relation, we have $1 + a_1 = 1$, *i.e.* $a_1 = 0$
So, $P(x) = x$

3. (c) : The given equation is $\sum_{i=1}^{n}\frac{P'(x)}{P(x)-x_i} = \frac{n^2}{x}$

Integrating, $\sum_{i=1}^{n}\log|P(x)-x_i| = n^2\log c|x|, c > 0$

or, $\log\prod_{i=1}^{n}|P(x)-x| = \log c^{n^2}|x|^{n^2}$

i.e. $\prod_{i=1}^{n}|P(x)-x_i| = k|x|^{n^2}, k > 0$

i.e. $|P(P(x)| = k|x|^{n2}$
Hence, $P(x) = ax^n$

4. (a) : Define, $Q(x) = xP(x)$.
As $a_n \ne 0$, the polynomial Q has distinct real roots. So, $Q'(x)$ has distinct real roots as well.

Define $f(x) = x\,Q'(x)$.

Again, we observe that H' has distinct real roots and since $H'(x) = x^2P''(x) + 3xP'(x) + P(x)$ has distinct real roots.

5. (a) : Numerator $= f((3+\sqrt{7})^n a)$

$$= f\left[(3+\sqrt{7})^n a + (3-\sqrt{7})^n a - (3-\sqrt{7})^n a\right]$$

$$= f[Ma - (3-\sqrt{7})^n a]$$

where $(3+\sqrt{7})^n + (3-\sqrt{7})^n = \text{integer } (M)$

$$= f(-(3-\sqrt{7})^n a) \text{ [as } f \text{ is periodic]}$$

Similarly, denominator $= g((2+\sqrt{2})^n b)$

$$= g(-(2-\sqrt{2})^n b)$$

Hence, given limit

$$= \frac{a}{b}\lim_{n\to\infty}\left(\frac{f(-(3-\sqrt{7})^n)}{(-(3-\sqrt{7})^n)} \times \frac{-(2-\sqrt{2})^n}{g(-(2-\sqrt{2})^n)} \times \frac{(3-\sqrt{7})^n}{(2-\sqrt{2})^n}\right)$$

$$= \text{zero as } \frac{3-\sqrt{7}}{2-\sqrt{2}} < 1$$

6. (a) : $\lim\limits_{n\to\infty} f(x_0 + nT_1 + T_2) - g(x_0 + nT_1 + T_2) = 0$

$\lim\limits_{n\to\infty} f(x_0 + T_2) - g(x_0 + nT_1) = 0$

i.e. $f(x_0 + T_2) = g(x + nT_1)$ ... (1)

But according to the question,

$$\lim_{n\to\infty} g(x_0 + nT_1) - f(x_0 + nT_1) = 0$$

So, $\lim\limits_{n\to\infty} g(x_0 + nT_1) - f(x_0) = 0$

i.e. $f(x_0 + T_2) = f(x_0)$ (By (1))

i.e. T_2 is period of $f(x)$ as well.

7. (b) : Let $[x] = a$, $\{x\} = b$ then $x = a + b$

and $f(x) = \dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}}$

Squaring: $(f(x))^2 = \dfrac{a+b+2\sqrt{ab}}{a+b} = 1 + \dfrac{\sqrt{ab}}{\left(\frac{a+b}{2}\right)}$

Using A.M. $\geq$ G.M., we have $f(x) \leq \sqrt{2}$

8. (b) : $\lim\limits_{\substack{x\to+\infty \\ x\to-\infty}} = 1$ (given)

$$f(x) = \frac{x^3(8-n^3) + mx^2}{\sqrt[3]{(8x^3+mx^2)^2} + nx\cdot\sqrt[3]{8x^3+mx^2} + n^2x^2}$$

$(8 - n^3)$ must be zero and then

$$f(x) = \frac{m}{\left(8+\frac{m}{x}\right)^{2/3} + 2\left(8+\frac{m}{x}\right)^{1/3} + 4}$$

As $|x| \to \infty$, $f(x) = \dfrac{m}{12} = 1$ (A.T.Q)

So, $m = 12$

9. (c) : $f(x) - f(x/2) = x/2$

$f(x/2) - f(x/4) = x/4$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$f\left(\frac{x}{2^{n-1}}\right) - f\left(\frac{x}{2^n}\right) = \frac{x}{2^n}$$

Adding and $n\to\infty$, $f(x) - f(0) = \dfrac{x}{2} + \dfrac{x}{4} + \ldots + \infty$

Hence, $f(x) = x + 1$

10. (b) : Putting $y = x - 1$ in L.H.S. and R.H.S, we have $f(x) \leq g(x)$.

Putting $y = x + 1$ in L.H.S. and R.H.S., $f(x) \geq g(x)$

Hence, for all x, $f(x) = g(x)$ and so inequality becomes equality and $(x - y)f(x) + h(x) - xy + y^2 = h(y)$

For $x = 0$, $h(y) = y^2 - y\,f(0) + h(0)$ for all $y \in R$.

Hence, $h(x)$ is a quadratic function.

11. (d) : Let $f(x_0) = 2$ then at $x = x_0$,

$f(f(x_0))\cdot(1 + f(x_0)) = -f(x_0)$

i.e. $f(2)(1 + 2) = -2$

Hence, $f(2) = \dfrac{-2}{3}$

12. (b) : Let $a_0 = \cos\theta$, $\theta \in (0, \pi)$

So, $a_1 = \sqrt{\dfrac{1+\cos\theta}{2}} = \cos\left(\dfrac{\theta}{2}\right)$

Similarly, $a_2, a_3, \ldots, a_n = \cos(\theta/2^n)$

Hence, $A_n = 4^n\left(1 - \cos\left(\dfrac{\theta}{2^n}\right)\right)$

$$= \frac{\theta^2}{1+\cos\left(\frac{\theta}{2^n}\right)}\left[\frac{\sin(\theta/2^n)}{\theta/2^n}\right]^2$$

So, as $n\to\infty$, $A_n = \dfrac{\theta^2}{2}(1) = \dfrac{\theta^2}{2} = \dfrac{1}{2}(\cos^{-1}a_0)^2$

Solution Sender of Maths Musing

SET-176

1. Devjit Acharjee (West Bengal)

13. (c) : The given limit = $\int_0^1 \left[\frac{2}{x}\right]-2\left[\frac{1}{x}\right]\cdot dx$

$$\sqrt{f(x)=\left[\frac{2}{x}\right]-2\left[\frac{1}{x}\right]}=\begin{cases}0 \text{ if } x\in\left(\frac{2}{2n+1},\frac{2}{2n}\right)\\ 1 \text{ if } x\in\left(\frac{2}{2n+2},\frac{2}{2n+1}\right)\end{cases}$$

So, required integral

$$=\left(\frac{2}{3}-\frac{2}{4}\right)+\left(\frac{2}{5}-\frac{2}{6}\right)+\left(\frac{2}{7}-\frac{2}{8}\right)+....$$

$$=2\left[\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-....\right]=2\left(\log 2-1+\frac{1}{2}\right)$$

$= \log 4 - 1$

14. (a) : Given inequation becomes

$$\frac{d}{dx}(x+\tan^{-1}f(x))=\frac{f'(x)}{1+f^2(x)}+1\geq 0$$

So, $x + \tan^{-1}f(x)$ is non-dec.

Hence, $\frac{\pi}{2}+a\leq-\frac{\pi}{2}+b$, so $b-a\geq\pi$

Notice that $a = 0$, $b = x$, $f(x) = \cot x$ gives the minimum condition.

15. (d) : $g(x) = f(2^x, 0) = f(2^{x+1}, -2^{x+1})$
$= f(0, -2^{x+3}) = f(-2^{x+4}, -2^{x+4})$
$= f(-2^{x+6}, 0) = f(-2^{x+7}, 2^{x+7})$
$= f(0, 2^{x+9}) = f(2^{x+10}, 2^{x+10})$
$= f(2^{x+12}, 0) = g(x + 12)$

Hence, $g(x)$ is periodic with period = 12

16. (a) : Let $f(x)=\frac{x}{\sqrt{x+1}}$. Domain $(-1, -\infty)$

Notice, $f'(x)>0$ and $f''(x)<0$. So, f is strictly increasing and concave in its domain.

Now, $f(3) = 3/2$ and $f'(3) = 5/16$. So, the tangent line to graph of f at (3, 3/2) is $y=\frac{3}{2}+\frac{5}{16}(x-3)$. Since, f is strictly concave, so $f(x)<\frac{3}{2}+\frac{5}{16}(x-3)$

Hence, the only solution is $x = 3$.

17. (c) : Let $g(x) = \log(1 + x + x^2)$. Domain is $x \in R$.

$$g''(x)<0 \Rightarrow x>\frac{\sqrt{3}-1}{2} \text{ or } <\frac{-(1+\sqrt{3})}{2}$$

So, f is concave, when $b<\frac{(1+\sqrt{3})}{2}$

and $\lim_{x\to b} f(x)=f(b)$ for continuity of f gives

$\log(1 + b + b^2) = ab + c$ and if f is concave then

$$f'(b^-)\geq f'(b^+) \Rightarrow \frac{2b+1}{1+b+b^2}\geq a$$

18. (a) : The subsequence of b_n corresponding to odd integers is $b_{2n+1} = (-1)^n(2n + 1)$, which does not have a fixed limit. Hence b_n also does not have a limit. The sequence c_n is product of bounded sequence a_n and $(1/n)$. So limit tends to zero.

And $d_n=a_n\cos\left(\frac{n\pi}{2}\right)=\frac{1}{2}\sin(n\pi)=0,\ n\in N.$

MATH archives

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE Main & Advanced Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE Main & Advanced. In every issue of MT, challenging problems are offered with detailed solution. The readers' & comments and suggestions regarding the problems and solutions offered are always welcome.

1. The eccentricity of the ellipse which meets the straight line $\frac{x}{7}+\frac{y}{2}=1$ on the axis of x and the straight line $\frac{x}{3}-\frac{y}{5}=1$ on the axis of y and whose axes lie along the axes of coordinates is

(a) $\frac{2\sqrt{6}}{7}$ (b) $\frac{3\sqrt{2}}{7}$ (c) $\frac{\sqrt{6}}{7}$ (d) $\frac{4\sqrt{2}}{7}$

2. If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then k is equal to

(a) 9/2 (c) 0 (d) –1 (d) 2/9

3. The vectors $\vec{a}$ and $\vec{b}$ are not perpendicular and $\vec{c}$ and $\vec{d}$ are the vectors satisfying $\vec{b}\times\vec{c}=\vec{b}\times\vec{d}$ and $\vec{a}\cdot\vec{d}=0$. Then the vector $\vec{d}$ is equal to:

(a) $\vec{c}-\left(\frac{\vec{a}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{b}$ (b) $\vec{b}-\left(\frac{\vec{b}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{c}$

(c) $\vec{c}+\left(\frac{\vec{a}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{b}$ (d) $\vec{b}+\left(\frac{\vec{b}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{c}$

4. The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only, is

(a) 55 (b) 66 (c) 77 (d) 88

5. The mean of two samples of sizes 200 and 300 were found to be 25, 10 respectively. Their standard deviations were 3 and 4 respectively. The variance of combined sample of size 500 is

(a) 64 (b) 65.2 (c) 67.2 (d) 64.2

6. The mean and variance of random variable X having a binomial distribution are 4 and 2 respectively, then $P(X = 1)$ is

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$ (c) $\frac{1}{4}$ (d) $\frac{1}{32}$

7. The only statement among the following that is a tautology is:

(a) $A\wedge(A\vee B)$ (b) $A\vee(A\wedge B)$

(c) $[A\wedge(A\to B)]\to B$ (d) $B\to[A\wedge(A\to B)]$

8. In a ΔPQR, if $3\sin P+4\cos Q=6$ and $4\sin Q+3\cos P=1$, Then the angle R is equal to

(a) $\pi/4$ (b) $3\pi/4$ (c) $5\pi/6$ (d) $\pi/6$

9. The equation $2\cos^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ is valid for all values of x satisfying

(a) $-1\le x\le 1$ (b) $0\le x\le 1$

(c) $0\le x\le 1/\sqrt{2}$ (d) $1/\sqrt{2}\le x\le 1$

10. Let $f(x) = (x - 4)(x - 5)(x - 6)$ then

(a) $f'(x) = 0$ has four real roots

(b) three roots of $f'(x) = 0$ lie in $(4,5)\cup(5,6)\ \cup(6,7)$

(c) the equation $f'(x)$ has only two roots

(d) three roots of $f'(x)$ lie in $(3,4)\cup(4,5)\cup(5,6)$

By : **Prof. Shyam Bhushan,** Director, Narayana IIT Academy, Jamshedpur. Mob. : 09334870021

SOLUTIONS

1. (a) : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(i)

$\frac{x}{7}+\frac{y}{2}=1$ meets x-axis at $A(7,0)$, line $\frac{x}{3}-\frac{y}{5}=1$ meets y-axis at $B(0,-5)$

Eq. (i) passes through A and B

$\Rightarrow \frac{49}{a^2}+0=1, 0+\frac{25}{b^2}=1$

$\Rightarrow a^2=49, b^2=25; b^2=a^2(1-e^2) \Rightarrow 25=49(1-e^2)$

$\Rightarrow e^2=\frac{24}{49} \Rightarrow e=\frac{2\sqrt{6}}{7}$

2. (a) : Any point on the first line is $(2r_1+1, 3r_1-1, 4r_1+1)$

And on the second line is $(r_2+3, 2r_2+k, r_2)$

The lines will intersect when

$2r_1+1=r_2+3, 3r_1-1=2r_2+k, 4r_1+1=r_2$

$\Rightarrow 2r_1-r_2=2, 4r_1-r_2=-1 \Rightarrow r_1=-3/2, r_2=-5$

And $k=3r_1-2r_2-1=-9/2+10-1=9/2$

3. (a) : $\vec{b}\times\vec{c}=\vec{b}\times\vec{d} \Rightarrow \vec{b}\times(\vec{c}-\vec{d})=0$

$\Rightarrow \vec{c}-\vec{d}\,||\,\vec{b} \Rightarrow \vec{c}-\vec{d}=\alpha\vec{b}$ for some $\alpha\in R \Rightarrow \vec{d}=\vec{c}-\alpha\vec{b}$

Also, $\vec{a}\cdot\vec{d}=\vec{a}\cdot\vec{c}-\alpha\vec{a}\cdot\vec{b} \Rightarrow 0=\vec{a}\cdot\vec{c}-\alpha\vec{a}\cdot\vec{b}$

$\Rightarrow \alpha=\frac{\vec{a}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}$ Thus, $\vec{d}=\vec{c}-\frac{\vec{a}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\vec{b}$

4. (c) : There are only two possibilities for sum of the digits equal to 10.

Case (i): 1,1,1,1,1,2,3

Number of seven digit integers $=\frac{7!}{5!}=42$

Case (ii): 1,1,1,1,2,2,2

Number of seven digit integers $=\frac{7!}{4!3!}=35$

$\therefore$ Total number of integers = 42 + 35 = 77

5. (c) : Combined mean $\bar{x}=\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}$

$=\frac{200\times25+300\times10}{500}=16$

Let $d_1=\bar{x}_1-\bar{x}=25-16=9, d_2=\bar{x}_2-\bar{x}=10-16=-6$

Now we know that $\sigma^2=\frac{n_1(\sigma_1^2+d_1^2)+n_2(\sigma_2^2+d_2^2)}{n_1+n_2}$

$=\frac{200(9+81)+300(16+36)}{500}=\frac{33600}{500}=67.2$

6. (d) : We are given, $np=4$, $npq=2$

$\Rightarrow q=\frac{2}{4}=\frac{1}{2} \therefore n\left(\frac{1}{2}\right)=4 \Rightarrow n=8$

$\therefore P(X=1)={}^nC_1pq^{n-1}=8pq^7=8\left(\frac{1}{2}\right)^8=\frac{1}{32}$

7. (c) : Note that $A\wedge(A\vee B)$ is F when $A=F$

$A\vee(A\wedge B)$ is F when $A=F$, $B=F$, and

$B\to[A\wedge(A\to B)]$ is F when $A=F$, $B=T$

$\therefore$ We check only (c)

$[A\wedge(A\to B)]\to B\equiv[A\wedge(\sim A\vee B)]\to B$

$\equiv[(A\wedge(\sim A))]\vee(A\wedge B)\to B$

$\equiv A\wedge B\to B\equiv\sim(A\wedge B)\vee B\equiv\sim[(A\wedge B)\wedge(\sim B)]$

$\equiv\sim[A\wedge(B\wedge\sim B)]\equiv\sim[A\wedge F]\equiv\sim F\equiv T$

Thus $[A\wedge(A\to B)]\to B$ is a tautology.

8. (d) : Squaring and adding the given relations we get $16+9+24\sin(P+Q)=37$

$\Rightarrow \sin(P+Q)=1/2 \Rightarrow \sin R=1/2$

$\Rightarrow R=\pi/6$ or $5\pi/6$

If $R=5\pi/6$, then $P<\pi/6$

$\Rightarrow 3\sin P<3/2 \Rightarrow 3\sin P+4\cos Q<3/2+4<6$

So, $R\neq5\pi/6$

9, (d) : If we denote $\cos^{-1}x$ by y, then since

$0\le\cos^{-1}x\le\pi \Rightarrow 0\le2y\le2\pi,$ ...(i)

Also since $-\pi/2\le\sin^{-1}\left(2x\sqrt{1-x^2}\right)\le\pi/2$

$\Rightarrow -\pi/2\le\sin^{-1}\sin(2y)\le\pi/2,$ ...(ii)

$-\frac{\pi}{2}\le2y\le\frac{\pi}{2}$

From (i) and (ii) we find

$0\le2y\le\pi/2 \Rightarrow 0\le y\le\pi/4 \Rightarrow 0\le\cos^{-1}x\le\pi/4$

Which holds if $1/\sqrt{2}\le x\le1$

10. (b) : Since $f(4)=f(5)=f(6)=f(7)=0$, so by Rolle's theorem applied to the intervals [4, 5], [5, 6], [6, 7] there exist $x_1\in(4,5), x_2\in(5,6), x_3(6,7)$ such that $f'(x_1)=f'(x_2)=f_1(x_3)=0$. Since f' is a polynomial of degree 3 so cannot have four roots.

◈◈

MATHS MUSING

SOLUTION SET-176

1. **(b) :** $\frac{a+b}{2} = \sqrt{ab} + 2 \Rightarrow \sqrt{b} = \sqrt{a} + 2$

$(a, b) = (1, 9), (4, 16), (9, 25), (16, 36), (25, 49), (36, 64), (49, 81), (64, 100), (81, 121) = 9$ pairs.

2. **(d) :** The n^{th} term, t_n

$$= \frac{1^3 + 2^3 + ... + n^3}{1 + 3 + ... + (2n-1)} = \frac{\frac{n^2(n+1)^2}{4}}{n^2} = \frac{(n+1)^2}{4}$$

$$\sum_{n=1}^{9} t_n = \sum_{n=1}^{9} \frac{(n+1)^2}{4} = \frac{1}{4}\left\{\sum_{n=1}^{10} n^2 - 1\right\}$$

$$= \frac{1}{4}\left\{\frac{10 \cdot 11 \cdot 21}{6} - 1\right\} = \frac{1}{4}\{385 - 1\} = \frac{1}{4} \times 384 = 96$$

3. **(a) :** The area of the pentagon

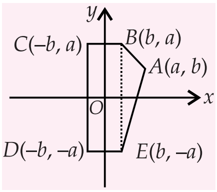
$=$ Area of $BCDE$ + Area of ΔEAB

$= 4ab + a^2 - ab$

$= a(3b + a) = 451 = 11 \times 41$

$\Rightarrow \; a = 11, b = 10$

$\therefore \; a - b = 1$



4. **(c) :** $y^2 = 4ax \Rightarrow yy' = 2a$

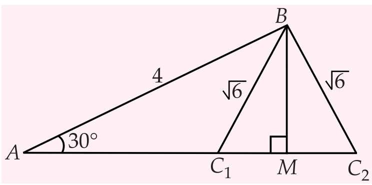
Eliminating a, $2x\,y' = y$

Replacing y' by $-\frac{1}{y'}$ we get, $2x + yy' = 0$

Solving, $x^2 + \frac{y^2}{2} = c, \; c \in R$

These are ellipses with eccentricity $= \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$

5. **(c) :**



$BM = 2$, $C_1M = \sqrt{6-4} = \sqrt{2} = MC_2$

$AM = 4\cos 30° = 2\sqrt{3} \;\therefore\; b = 2\sqrt{3} + \sqrt{2}$

Area $= \frac{1}{2} bc \sin A = b = 2\sqrt{3} + \sqrt{2}$

6. **(b, c, d) :** $\left(\frac{1}{2}\right)^{2(x-1)/(2x^2+x-1)} < \left(\frac{1}{2}\right)^{1/x}$

$$\Rightarrow \; \frac{2(x-1)}{2x^2 + x - 1} > \frac{1}{x} \;\Rightarrow\; \frac{1-3x}{x(2x-1)(x+1)} > 0$$

$\Rightarrow \; (1 - 3x)x(2x - 1)(x + 1) > 0$

$$\Rightarrow \; x(x+1)\left(x - \frac{1}{3}\right)\left(x - \frac{1}{2}\right) < 0$$

$$\therefore \; x \in (-1, 0) \cup \left(\frac{1}{3}, \frac{1}{2}\right)$$

7. **(a) :** $z_1 = \cos\theta + i\sin\theta$

$z_2 = \cos 2\theta + i\sin 2\theta - (\cos\theta + i\sin\theta)$

$= (\cos 2\theta - \cos\theta) + i(\sin 2\theta - \sin\theta)$

$|z_2|^2 = (\cos 2\theta - \cos\theta)^2 + (\sin 2\theta - \sin\theta)^2$

$= 2 - 2(\cos 2\theta \cos\theta + \sin 2\theta \sin\theta) = 2 - 2\cos\theta = 4\sin^2 \frac{\theta}{2}$

$\therefore \; |z_2| = 2\left|\sin\frac{\theta}{2}\right|$

8. **(c) :** $z_2 = (\cos 2\theta - \cos\theta) + i(\sin 2\theta - \sin\theta)$

$$= -2\sin\frac{3\theta}{2}\sin\frac{\theta}{2} + i\,2\cos\frac{3\theta}{2}\sin\frac{\theta}{2}$$

$$= 2i\sin\frac{\theta}{2}\left(\cos\frac{3\theta}{2} + i\sin\frac{3\theta}{2}\right)$$

$$\arg z_2 = \arg\left(2i\sin\frac{\theta}{2}\right) + \arg\left(\cos\frac{3\theta}{2} + i\sin\frac{3\theta}{2}\right) = \frac{\pi}{2} + \frac{3\theta}{2}$$

Since, $4n\pi < \theta < (4n + 2)\pi$

$$\therefore \; 2n\pi < \frac{\theta}{2} < (2n+1)\pi \;\Rightarrow \sin\frac{\theta}{2} > 0$$

9. **(5) :** $m = 2^{2013} = 4 \cdot 2^{2011}$

$2^m - 1 = (2^4)^{2^{2011}} - 1 = 16^{2 \cdot 2011} - 1$

Last digit is $6 - 1 = 5$

10. **(b) : P.** $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x}$

$$= \lim_{x \to 0} x^{\alpha - 1} \sin\left(\frac{1}{x}\right) = 0 \text{ if } \alpha > 1$$

$$x \neq 0 \Rightarrow f'(x) = \alpha x^{\alpha - 1}\sin\left(\frac{1}{x}\right) - x^{\alpha - 2}\cos\left(\frac{1}{x}\right)$$

$\lim_{x \to 0} f'(x) = 0$ if $\alpha > 2 \quad \therefore \; \alpha = 3$

Q. $y(1) = 2 \Rightarrow 2 = a + b + c$, $y(0) = 0 \Rightarrow c = 0$

Also, $y'(0) = 1 \Rightarrow b = 1 \;\therefore\; a = 1$

Hence, $y = x^2 + x \Rightarrow y(-1) = 0$

R. $f(x) = \sin x(1 + a)$, where $a = \int_0^{\pi/2} \cos t\,dt = 1$

$$\therefore \; f(x) = 2\sin x, \; \int_0^{\pi/2} f(x)dx = 2.$$

S. The hyperbolas $\frac{x^2}{2} - y^2 = 1$ and $\frac{y^2}{2} - x^2 = 1$ have common tangents with slopes ± 1. The four tangents are $x + y \pm 1 = 0$, $x - y \pm 1 = 0$. They form a square of area $\sqrt{2} \times \sqrt{2} = 2$

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